Pythagorean Identities

$$\sin^2 x = 1 - \cos^2 x$$

$$\cos^2 x = 1 - \sin^2 x$$

$$\tan^2 x = \sec^2 x - 1$$

$$\sec^2 x = 1 + \tan^2 x$$

Angle Sum Formulas

$$\sin^2 x = \frac{1}{2}[1 - \cos 2x]$$

$$\cos^2 x = \frac{1}{2}[1 + \cos 2x]$$

$$\sin(mx)\sin(nx) = \frac{1}{2}[\cos(m-n)x - \cos(m+n)x]$$

$$\cos(mx)\cos(nx) = \frac{1}{2}[\cos(m-n)x + \cos(m+n)x]$$

$$\sin(mx)\cos(nx) = \frac{1}{2}[\sin(m-n)x + \sin(m+n)x]$$

$$\sin 2x = 2\sin x \cdot \cos x$$

$$\cos 2x = 2\cos^2 x - 2$$

Products of Sines and Cosines

$$\int \sin(mx)\sin(nx)\,dx$$

$$\int \sin(mx)\cos(nx)\,dx$$

$$\int \cos(mx)\cos(nx)\,dx$$

$$(m \neq n)$$

Ex:

$$\int \sin(3x)\sin(4x)\,dx$$

$m=3 \qquad n=4$

$$= \int \frac{1}{2}\left[\cos(-x) - \cos(7x)\right] dx$$

$$= \frac{1}{2}\int \cos(-x)\,dx - \frac{1}{2}\int \cos(7x)\,dx$$

$u=-x \qquad u=7x$

$du=-dx \qquad du=7dx$

$$= -\frac{1}{2}\int \cos u\,du - \frac{1}{14}\int \cos u\,du$$

$$= -\frac{1}{2}\sin u - \frac{1}{14}\sin u + C$$

$$= -\frac{1}{2}\sin(-x) - \frac{1}{14}\sin(7x) + C$$

8.3 Trigonometric Substitutions

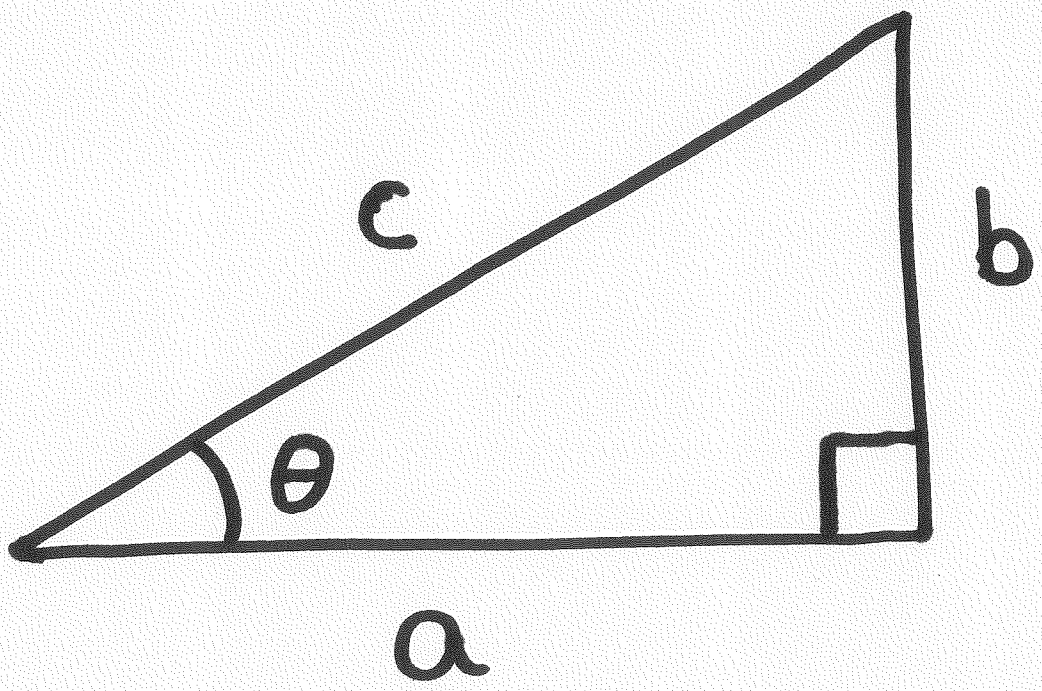
Idea: Trig. Identities are very flexible, especially in completing square roots.

Alert:

$$\sqrt{a^2 + x^2}$$

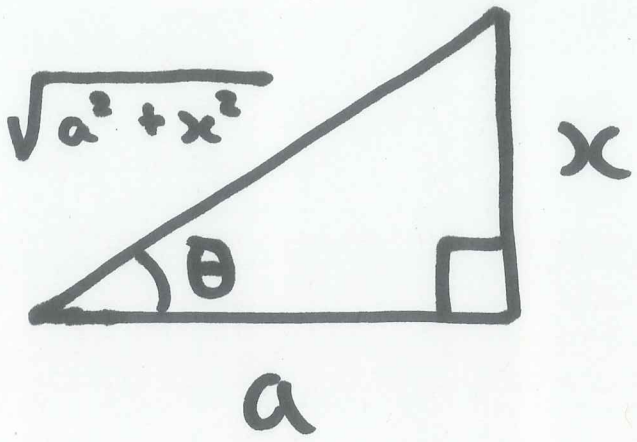
$$\sqrt{a^2 - x^2}$$

$$\sqrt{x^2 - a^2}$$



$$a^2 + b^2 = c^2$$

SOH CAH TOA



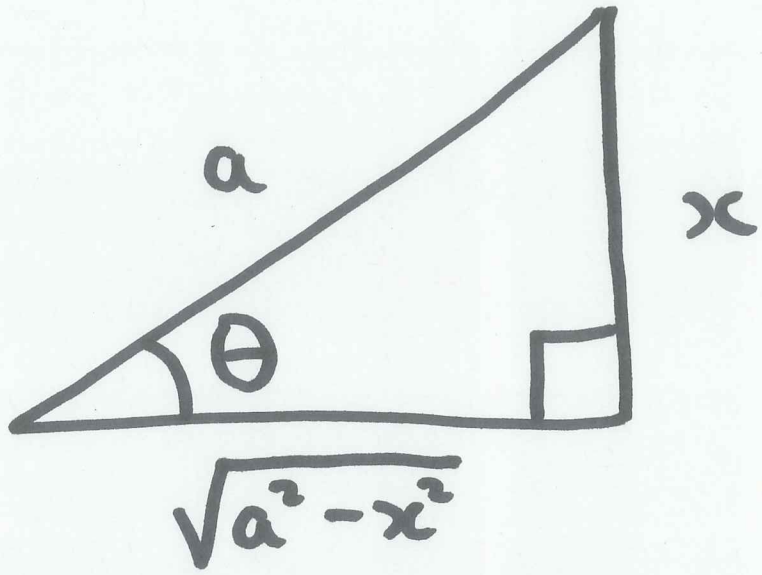
$$x = a\tan\theta$$

$$\sqrt{a^2+x^2} = a|\sec\theta|$$

$$\sec\theta = \frac{1}{a}\sqrt{a^2+x^2}$$

$$\cos\theta = \frac{a}{\sqrt{a^2+x^2}}$$

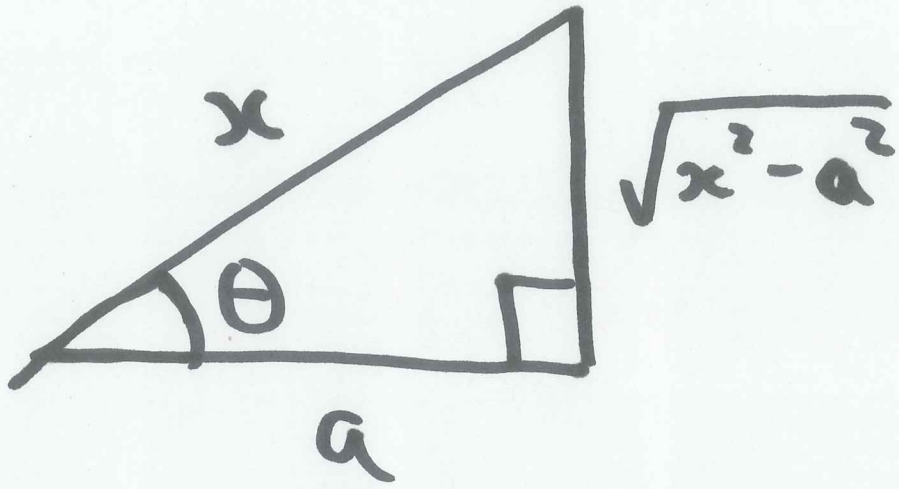
SOH CAH TOA



$$x = a \sin\theta$$

$$\sqrt{a^2-x^2} = a\,|\cos\theta|$$

SOH CAH TOA



$$x = a \sec\theta = \frac{a}{\cos\theta}$$

$$\sqrt{x^2-a^2} = a|\tan\theta|$$

BE CAREFUL OF BOUNDS

$x = a\tan\theta \rightarrow \theta = \arctan\left(\frac{x}{a}\right)$

$$-\frac{\pi}{2} < \theta < \frac{\pi}{2}$$

$x = a\cdot\sin\theta \rightarrow \theta = \arcsin\left(\frac{x}{a}\right)$

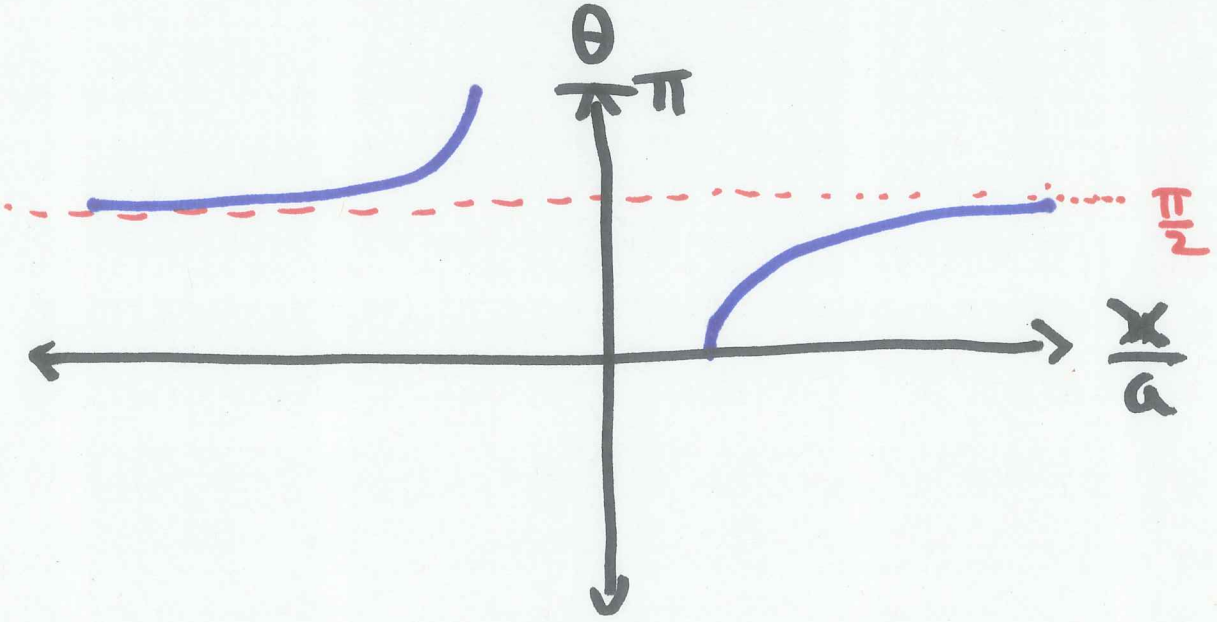
$$-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$$

$x = a\cdot\sec\theta \rightarrow \theta = \operatorname{arcsec}\left(\frac{x}{a}\right)$

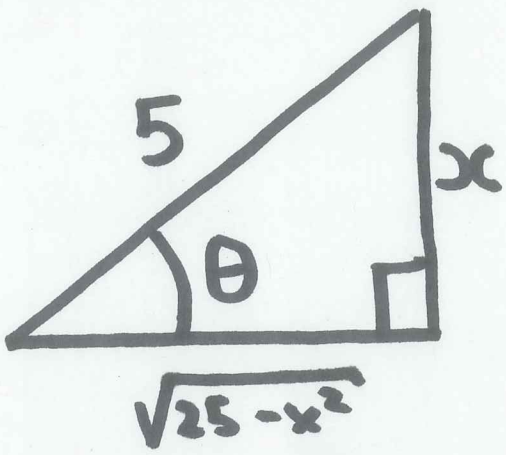
$$0 \leq \theta < \frac{\pi}{2} \quad \text{if } \frac{x}{a} \geq 1$$

-OR-

$$\frac{\pi}{2} < \theta \leq \pi \quad \text{if } \frac{x}{a} \leq -1$$



Ex $\int \sqrt{25 - x^2}\, dx$



$x = 5\sin\theta$

$\sqrt{25-x^2} = 5|\cos\theta|$

$x = 5\sin\theta$

$dx = 5\cos\theta\, d\theta$

$-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$

$$= \int \underbrace{5|\cos\theta|}_{\sqrt{25-x^2}} \cdot \underbrace{5\cos\theta\, d\theta}_{dx}$$

$$= 25\int \cos^2\theta\, d\theta$$

Half Angle Formula

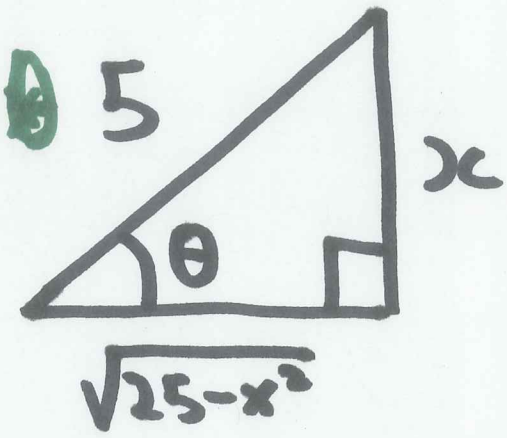
$$= 25\int \tfrac{1}{2}[1 + \cos 2\theta]\, d\theta$$

$$= \frac{25}{2}\theta + \frac{25}{4}\sin(2\theta) + C$$

$$= \frac{25}{2}\theta + \frac{25}{4}\sin(2\theta) + C$$

$\theta = \arcsin\left(\frac{x}{5}\right)$

$\sin 2\theta = 2\sin\theta\cos\theta$



$$= \frac{25}{2}\arcsin\left(\frac{x}{5}\right) + \frac{25}{2}\sin\theta\cdot\cos\theta + C$$

$$\left(\sin 2\theta = 2\sin\theta\cos\theta\right)$$

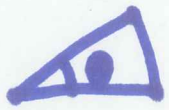
$$\sin\theta = \frac{x}{5} \qquad \cos\theta = \frac{\sqrt{25-x^2}}{5}$$

$$= \frac{25}{2}\arcsin\left(\frac{x}{5}\right) + \frac{25}{2}\left(\frac{x}{5}\right)\left(\frac{\sqrt{25-x^2}}{5}\right) + C$$

$$= \frac{25}{2}\arcsin\left(\frac{x}{5}\right) + \frac{1}{2}\times\sqrt{25-x^2} + C$$

Procedure for Trig. Subst.

1. Write substitution for x.
   Calculate dx.
   Specify Values of θ.

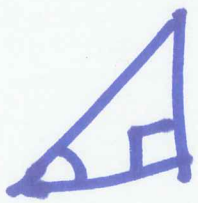


8.2

2. Substitute the trig. expression and simplify.

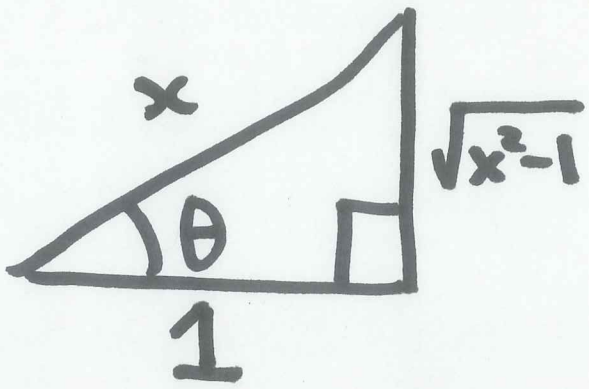
3. Integrate, being careful about θ Values.

4. Use reference triangle to convert back to x.



Ex: $\int \frac{dx}{x^2\sqrt{x^2-1^2}}$

$a=1$



$x = 1 \cdot \sec\theta$

$\sqrt{x^2-1} = 1 \cdot |\tan\theta|$

$dx = \sec\theta \cdot \tan\theta \, d\theta$

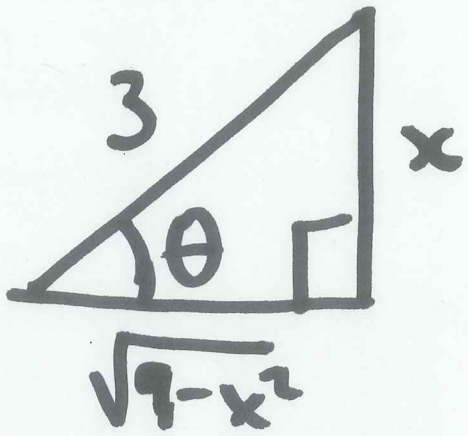
Ex:

$$\int \frac{x\,dx}{\sqrt{\underbrace{9-x^2}_{u}}} \quad \text{b·du}$$

$u = 9 - x^2$

$du = -2x\,dx$

$$= -\frac{1}{2}\int \frac{du}{\sqrt{u}}$$



$x = 3 \cdot \sin\theta$

$\sqrt{9-x^2} = 3 \cdot |\cos\theta|$

$dx = 3\cos\theta\,d\theta$

Ex: $\int \sqrt{\frac{4-x}{x}}\, dx$

(Hint: $u^2 = x$)