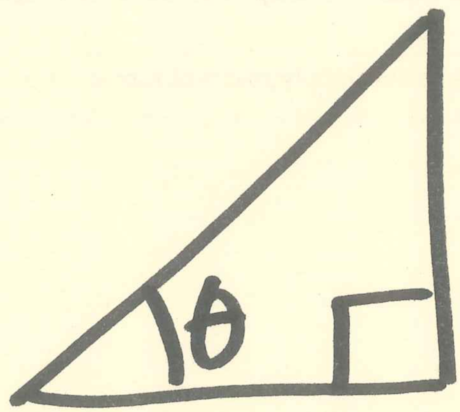
Reference Triangle



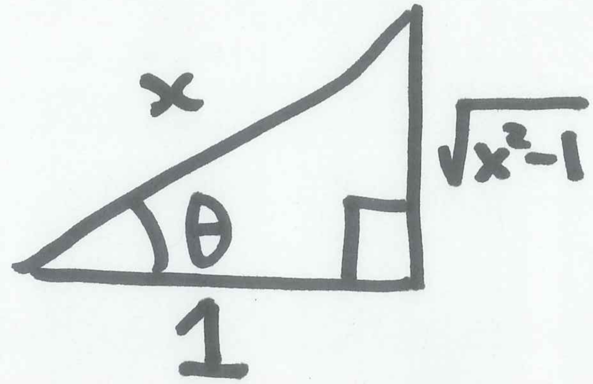
x : opposite or hypotenuse

a : adjacent or hypotenuse

$\sqrt{*}$: whatever remains

Ex: $\displaystyle\int \frac{dx}{x^2\sqrt{x^2-1^2}}$

$a=1$



$0 \leq \theta < \frac{\pi}{2}$ $\quad \neq$

$x = 1 \cdot \sec\theta$

$\sqrt{x^2-1} = 1 \cdot |\tan\theta|$

$dx = \sec\theta \cdot \tan\theta \, d\theta$

$\sin\theta = \dfrac{\sqrt{x^2-1}}{x}$

$$= \int \frac{\cancel{\sec\theta}\,\cancel{\tan\theta}\,d\theta}{\sec^{\cancel{2}}\theta\,\cancel{\tan\theta}}$$

$$= \int \frac{d\theta}{\sec\theta}$$

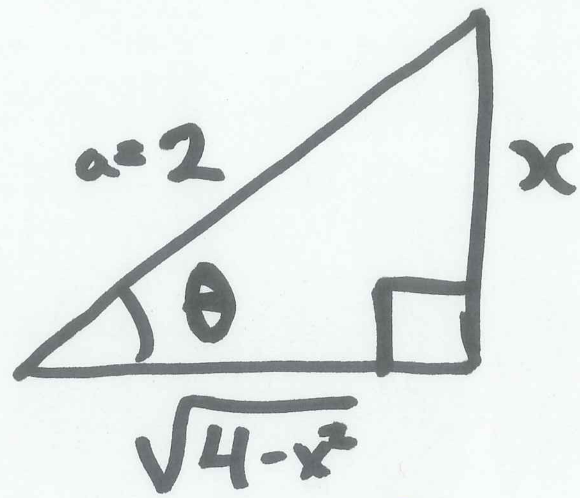
$$= \int \cos\theta \, d\theta$$

$$= \sin\theta + C = \boxed{\frac{\sqrt{x^2-1}}{x} + C}$$

back-Substitution!

Ex:

$$\int \frac{dx}{(4-x^2)^{3/2}}$$



$$= \int \frac{dx}{\left(\sqrt{4-x^2}\right)^3}$$

$$x = 2\sin\theta$$

$$\sqrt{4-x^2} = 2|\cos\theta|$$

$$-\frac{\pi}{2} < \theta < \frac{\pi}{2}$$

$$dx = 2\cos\theta\, d\theta$$

$$= \int \frac{2\cos\theta\, d\theta}{(2\cos\theta)^3}$$

$$= \int \frac{d\theta}{4\cos^2\theta}$$

$$= \frac{1}{4}\int \frac{d\theta}{\cos^2\theta}$$

$$= \frac{1}{4}\int \sec^2\theta\, d\theta$$

$$= \frac{1}{4}\tan\theta\, \cancel{d\theta} + C'$$

$$\boxed{= \frac{1}{4}\cdot\frac{x}{\sqrt{4-x^2}} + C'}$$

Ex: $\displaystyle\int \frac{dy}{y\sqrt{1+(\ln y)^2}}$

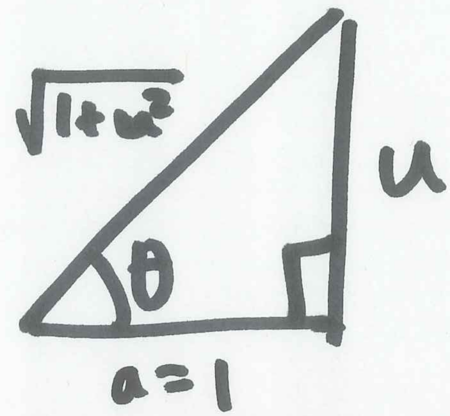
du

u

$u = \ln y$

$du = \dfrac{dy}{y}$

$$= \int \frac{du}{\sqrt{1+u^2}}$$



$-\frac{\pi}{2} < \theta < \frac{\pi}{2}$

$u = 1 \cdot \tan\theta$

$\sqrt{1+u^2} = 1 \cdot |\sec\theta|$

$du = \sec^2\theta \, d\theta$

$$= \int \frac{\sec^2\theta \, d\theta}{\sec\theta}$$

Memorize!

$$= \int \sec\theta \, d\theta$$

$$= \ln|\sec\theta + \tan\theta| + C$$

$$= \ln\left|\sqrt{1+u^2} + u\right| + C = \ln\left|\sqrt{1+\ln y^2} + \ln y\right| + C$$

Ex: $\displaystyle\int \sqrt{\frac{4-x}{x}}\,dx$

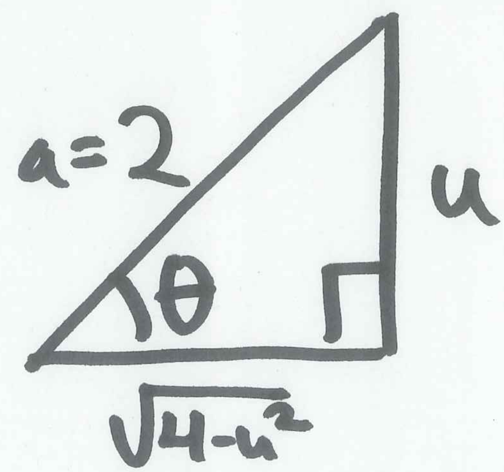
(Hint: $u^2 = x$)

$2u\,du = dx$

$u > 0$

$$= \int \frac{\sqrt{4-u^2}}{\cancel{\sqrt{u^2}}}\, 2\cancel{u}\,du$$

$$= \int 2\sqrt{4-u^2}\,du$$



⋮

Ex:

$$\int \frac{\sqrt{x-2}}{\sqrt{x-1}}\,dx \qquad x \geq 2$$

$$= \int \frac{\sqrt{(x-1)-1}}{\sqrt{x-1}}\,dx$$

Idea: Cancel one square root, leaving one for trig. subst.

$u^2 = x-1$

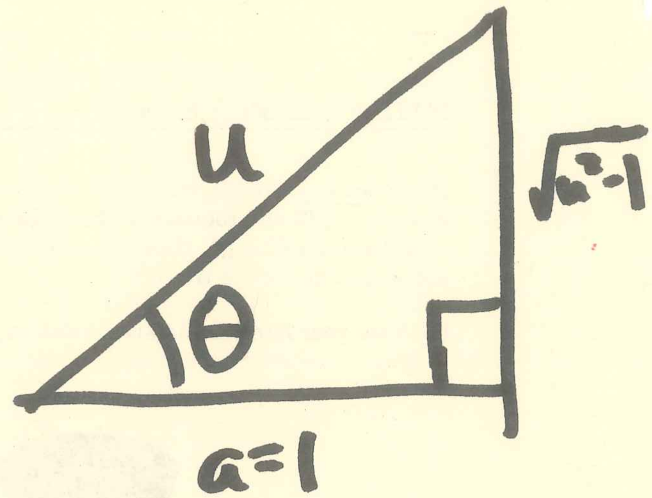
$u^2+1 = x$

$2u\,du = dx \qquad u \geq 1$

$$= \int \frac{\sqrt{u^2-1}}{\cancel{\sqrt{u^2}}}\,2\cancel{u}\,du$$

$$= \int 2\sqrt{u^2-1}\,du$$

$$= \int 2\sqrt{u^2-1}\,du$$



$0 \le \theta < \frac{\pi}{2}$

$u = 1 \cdot \sec\theta$

$\sqrt{u^2-1} = 1 \cdot |\tan\theta|$

$du = \sec\theta \cdot \tan\theta \, d\theta$

$$= \int 2\tan\theta \cdot \sec\theta \tan\theta \, d\theta$$

$$= \int 2\tan^2\theta \sec\theta \, d\theta$$

Try: $\tan^2\theta = \sec^2\theta - 1$

$$= 2\int (\sec^2\theta - 1)\sec\theta \, d\theta$$

$$= 2\int \sec^3\theta \, d\theta - 2\int \sec\theta \, d\theta$$

$$\int \sec^3\theta \, d\theta$$

in-class

Ex:

$$\int \frac{(1-x^2)^{1/2}}{x^4}\,dx$$