8.4: Partial Fractions

When: $\int \frac{f(x)}{g(x)}\,dx$ where

- $f(x)$, $g(x)$ are polynomials
- $f(x) \neq g'(x)$ (u-subst. instead)

How: 1. Factor $g(x)$

2. Split $f(x)$

3. Integrate Remaining

(lots of ln's & arctans)

Ex: $\displaystyle\int \frac{x^2\,dx}{x^4-1}$

$f(x) = x^2$ $\quad y = x^2$

$g(x) = x^4 - 1 = y^2 - 1 = (y+1)(y-1)$

$= (x^2+1)(x^2-1)$

$= (x^2+1)(x-1)(x+1)$

Roots: $1, -1$

$g(x) = (x-1)^1 (x+1)^1 (x^2+1)$

$$\frac{f(x)}{g(x)} = \frac{A_1}{x-1} + \frac{A_2}{x+1} + \frac{B_3 x + A_3}{x^2+1}$$

$$f(x) = A_1(x+1)(x^2+1) + A_2(x-1)(x^2+1) + (B_3x + A_3)\cdot(x-1)(x+1)$$

$$= A_1x^3 + A_1x^2 + A_1x + A_1 + A_2x^3 - A_2x^2 + A_2x - A_2 \cdots$$

$$f(x) = A_1 x^3 + A_1 x^2 + A_1 x + A_1$$

$$+ A_2 x^3 - A_2 x^2 + A_2 x - A_2$$

$$+ B_3 x^3 - B_3 x$$

$$+ A_3 x^2 - A_3$$

coeffs from $f(x) = x^2$

x^3: $0 = A_1 + A_2 + B_3$

x^2: $1 = A_1 - A_2 + A_3$

x^1: $0 = A_1 + A_2 - B_3$

x^0: $0 = A_1 - A_2 - A_3 = \frac{1}{4} - \frac{-1}{4} - A_3 = 0$

sum = sum

$$1 = 4A_1 \rightarrow A_1 = \frac{1}{4}$$

x^3: $0 = \frac{1}{4} + A_2 + B_3$

x^1: $0 = \frac{1}{4} + A_2 - B_3$

$+$ $-\frac{1}{2} = 2A_2 \longrightarrow A_2 = \frac{-1}{4}$ $\rightarrow B_3 = 0$, $\rightarrow A_3 = 1/2$

$$\frac{f(x)}{g(x)} = \frac{1/4}{x-1} + \frac{-1/4}{x+1} + \frac{1/2}{x^2+1}$$

$$\int \frac{x^2\,dx}{x^4-1} = \underbrace{\frac{1}{4}\int \frac{dx}{x-1}} - \underbrace{\frac{1}{4}\int \frac{dx}{x+1}} + \underbrace{\frac{1}{2}\int \frac{dx}{x^2+1}}$$

$$= \frac{1}{4}\ln|x-1| - \frac{1}{4}\ln|x+1| + \frac{1}{2}\arctan(x) + C$$

Ex: $\displaystyle\int \frac{e^t\,dt \leftarrow du}{e^{2t}+3e^t+2}$

$u = e^t$

$du = e^t\,dt$

$$= \int \frac{1\cdot du}{u^2+3u+2}$$

$f(u) = 1$

$g(u) = u^2+3u+2$

$= (u+2)(u+1)$

$$\frac{f(u)}{g(u)} = \frac{A_1}{u+2} + \frac{A_2}{u+1}$$

$f(u) = 1 = A_1(u+1) + A_2(u+2)$

$= A_1 u + A_1$

$A_2 u + 2A_2$

$u^1: \; 0 = A_1 + A_2 \rightarrow A_1 = -A_2 = -1$

$u^0: \; 1 = A_1 + 2A_2 \rightarrow 1 = -A_2 + 2A_2$

$\hookrightarrow 1 = A_2$

$$\int \frac{1}{u^2+3u+2}\,du = \int \frac{-1}{u+2}\,du + \int \frac{+1}{u+1}\,du$$

A_1 A_2

$$= -\int \frac{du}{u+2} + \int \frac{du}{u+1}$$

$$= -\ln|u+2| + \ln|u+1| + C$$

$$\underline{= -\ln|e^t+2| + \ln|e^t+1| + C}$$

Ex: $\displaystyle\int \frac{9x^3-3x+1}{x^3-x^2}\,dx = \int 9\,dx + \int \frac{9x^2-3x+1}{x^3-x^2}\,dx$

$$\begin{array}{r}
9 \\
x^3-x^2 \overline{\big)\, 9x^3+0x^2-3x+1} \\
-\underline{(9x^3-9x^2)} \\
9x^2-3x+1
\end{array}$$

$\leftarrow$ remainder!

$$\int \frac{9x^2-3x+1}{x^3-x^2}\,dx$$

$f(x) = 9x^2-3x+1$

$g(x) = x^3-x^2$

$= (x)^2(x-1)$

$\uparrow$ mult. 2!

$$\frac{f(x)}{g(x)} = \frac{A_1}{x} + \frac{A_2}{x^2} + \frac{A_3}{x-1}$$

$$f(x) = 9x^2 - 3x + 1 = A_1 x(x-1) + A_2(x-1) + A_3 x^2$$

$$= A_1 x^2 - A_1 x + A_2 x - A_2 + A_3 x^2$$

$$\begin{aligned}
x^2 &: 9 = A_1 + A_3 \to 9 = 2 + A_3 \\
x^1 &: -3 = -A_1 + A_2 \to -3 = -A_1 - 1 \\
x^0 &: 1 = -A_2 \to A_2 = -1
\end{aligned}$$

$$A_1 = 2$$

$$A_3 = 7$$

$$\frac{f(x)}{g(x)} = \frac{2}{x} + \frac{-1}{x^2} + \frac{7}{x-1}$$

... FINISH!

Ex: $\int \frac{x^6+1}{x^4-1}\,dx$

Step 1: Long Division
to reduce numerator

Ex:

$$\int \frac{1}{x^{3/2} - \sqrt{x}}\, dx$$

Alert:

$$\int \frac{Bx + A}{x^2 + ax + b}\,dx$$

if both $B, A \neq 0$, then trig subst becomes hard!

$$\int \underbrace{\frac{Bx + aB/2}{x^2+ax+b}\,dx}_{\substack{u = x^2+ax+b \\ du = (2x+a)\,dx}} + \int \underbrace{\frac{A - aB/2}{x^2+ax+b}\,dx}_{\text{trig subst.}}$$

$$\int \frac{B}{2u}\,du \rightsquigarrow \frac{B}{2}\ln|u| \ldots$$

Alert:

$$x^2 + ax + b$$

linear term is problematic!

$$= \left(x + \frac{a}{2}\right)^2 + \left(b - \frac{a^2}{4}\right)$$

$u = x + \frac{a}{2}$

$du = dx$

$$= \underbrace{u^2 + \left(b - \frac{a^2}{4}\right)}_{\text{look for arctan!}}$$

Ex:

$$\int \frac{x^2 - x + 2}{x^3 - 1}\,dx$$

$f(x) = x^2 - x + 2$

$g(x) = x^3 - 1$

Roots of g: 1

$g(x) = (x-1)(x^2 + x + 1)$

$$\begin{array}{r|l} & x^2 + x + 1 \\ \hline x - 1 & x^3 + 0x^2 + 0x - 1 \\ & -(x^3 - x^2) \\ \hline & \quad\; x^2 + 0x - 1 \\ & \quad -(x^2 - x) \\ \hline & \qquad\quad x - 1 \end{array}$$

$\overset{a}{x^2} + \overset{b}{x} + \overset{c}{1}$:

Discriminant $b^2 - 4ac$

$= 1^2 - 4 \cdot 1 \cdot 1$

$= -3 \leftarrow$ Negative!

IRREDUCIBLE

$$f(x) = x^2 - x + 2$$
$$g(x) = (x-1)(x^2+x+1)$$

$$\frac{f(x)}{g(x)} = \frac{A_1}{x-1} + \frac{B_2 x + A_2}{x^2+x+1}$$

$$f(x) = A_1 \frac{g(x)}{x-1} + (B_2 x + A_2)\frac{g(x)}{x^2+x+1}$$
$$= (A_1 x^2 + A_1 x + A_1) + (B_2 x^2 - B_2 x + A_2 x - A_2)$$
$$= (A_1 + B_2)x^2 + (A_1 - B_2 + A_2)x + (A_1 - A_2)$$

$x^2 : \; 1 = A_1 + B_2$

$x : \; -1 = A_1 - B_2 + A_2$

$1 : \; 2 = A_1 \qquad - A_2$

$x^2+x+1 : \; 2 = 3A_1 + 0B_2 + 0A_2$

$A_1 = \frac{2}{3} \longrightarrow B_2 = \frac{1}{3} \longrightarrow A_2 = \frac{-4}{3}$

$$\int \frac{x^2 - x + 2}{x^3 - 1}\,dx = \int \frac{2/3}{x-1}\,dx + \int \frac{\frac{1}{3}x - \frac{4}{3}}{x^2+x+1}\,dx$$

$$\frac{2}{3}\int \frac{dx}{x-1} = \frac{2}{3}\ln|x-1| + C$$

$$\frac{1}{3}\int \frac{x-4}{\underbrace{x^2+x+1}_{\text{want u-subst.}}}\,dx = \frac{1}{6}\int \frac{2x+1}{x^2+x+1} + \frac{1}{6}\int \frac{-9}{x^2+x+1}\,dx$$

$(2x+1)$

$2x-8 = (2x+1) - 9$

$$-\frac{3}{2}\int \frac{dx}{\underbrace{x^2+x}_{\text{complete the square}}+1}$$

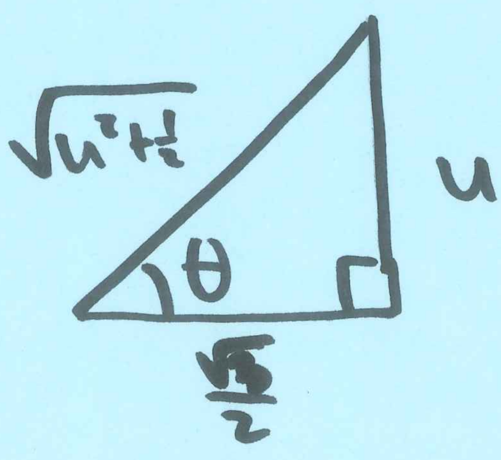
$$x^2 + x + \underset{\uparrow \frac{1}{2}}{\_\_} = \left(x + \underset{\uparrow \frac{1}{2}}{\_\_}\right)^2$$

$$= -\frac{3}{2}\int \frac{dx}{\left(x+\frac{1}{2}\right)^2+\frac{3}{4}}$$

$u = x + \frac{1}{2}$

$du = dx$

$$= -\frac{3}{2}\int \frac{du}{u^2+\frac{3}{4}}$$



$u = \frac{\sqrt{3}}{2}\tan\theta$

$du = \frac{\sqrt{3}}{2}\sec^2\theta$

$\sqrt{u^2+\frac{3}{4}} = \frac{\sqrt{3}}{2}\sec\theta$

$$-\frac{3}{2}\int \frac{du}{u^2+\frac{3}{4}} = -\frac{3}{2}\int \frac{\frac{\sqrt{3}}{2}\sec^2\theta \, d\theta}{\left(\frac{\sqrt{3}}{2}\sec\theta\right)^2}$$

$$= -\frac{3}{2}\int \frac{2\sqrt{3}}{3}\, d\theta$$

$$= -\sqrt{3}\,\theta + C$$

$$= -\sqrt{3}\arctan\left(\frac{2\sqrt{3}}{3}u\right) + C$$

$$= -\sqrt{3}\arctan\left(\frac{2\sqrt{3}}{3}\left(x+\frac{1}{2}\right)\right) + C$$

$$= -\sqrt{3}\arctan\left(\frac{2x+1}{\sqrt{3}}\right) + C$$

$$\int \frac{x^2 - x + 2}{x^3 - 1}\,dx$$

$$= \frac{2}{3}\int \frac{dx}{x-1} + \frac{1}{6}\int \frac{2x+1}{x^2+x+1}\,dx + \frac{1}{6}\int \frac{-9\,dx}{x^2+x+1}$$

$$\boxed{= \frac{2}{3}\ln|x-1| + \frac{1}{6}\ln|x^2+x+1| - \sqrt{3}\arctan\left(\frac{2x+1}{\sqrt{3}}\right) + C}$$