8.7: Improper Integrals

FTC: $\int_a^b f(x)\,dx = F(b) - F(a)$

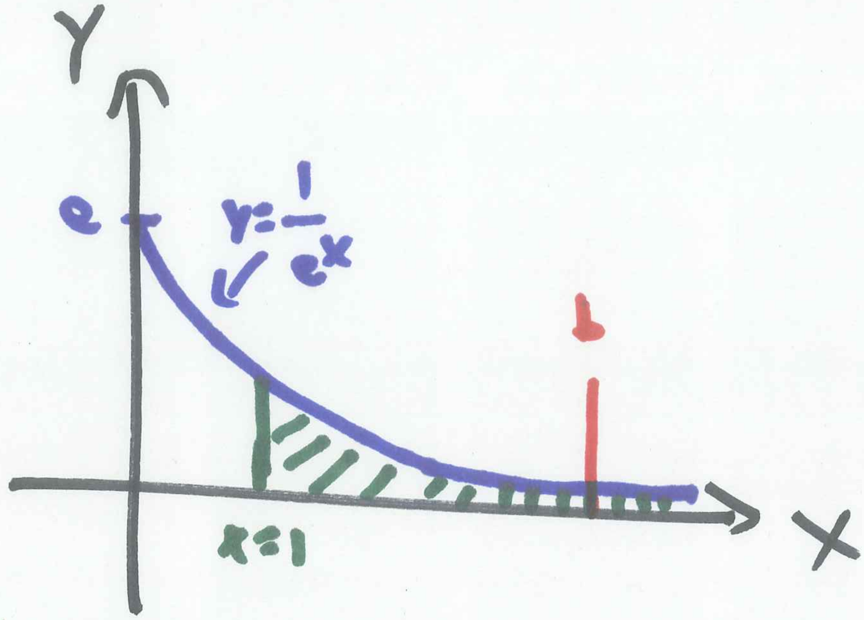
Requires: a, b finite

f cts. on $[a,b]$

F diff. on $[a,b]$

?: $\int_1^\infty e^{-x}\,dx = ?$

$$\int_1^\infty e^{-x}\,dx$$



$$\lim_{b\to\infty} \int_1^b e^{-x}\,dx$$

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$$\int e^{-x}\,dx = \underbrace{-e^{-x}}_{F(x)} + C$$

$$\int_1^b e^{-x}\,dx = \underbrace{-e^{-b}}_{F(b)} \underbrace{-}_{-} \underbrace{(-e^{-1})}_{F(1)}$$

$$= \frac{1}{e} - e^{-b}$$

$$\lim_{b\to\infty}\left(\frac{1}{e} - e^{-b}\right) = \frac{1}{e} - 0 \boxed{= \frac{1}{e}}$$

Improper Integral Type I:

1. If $f(x)$ is cts on $[a, \infty)$, then

$$\int_a^{\infty} f(x)\,dx = \lim_{b\to\infty} \int_a^b f(x)\,dx$$

2. If $f(x)$ is cts on $(-\infty, b]$, then

$$\int_{-\infty}^{b} f(x)\,dx = \lim_{a\to-\infty} \int_a^b f(x)\,dx$$

3. If $f(x)$ is cts on $(-\infty, \infty)$, then

$$\int_{-\infty}^{\infty} f(x)\,dx = \lim_{b\to\infty} \int_c^b f(x)\,dx + \lim_{a\to-\infty} \int_{-\infty}^{c} f(x)\,dx$$

for <u>any</u> $c \in \mathbb{R}$.

Improper Integrals can

CONVERGE

to a finite limit

or

DIVERGE

to an infinite limit

or no limit at all!

see $\int_0^\infty \sin(x)\,dx$

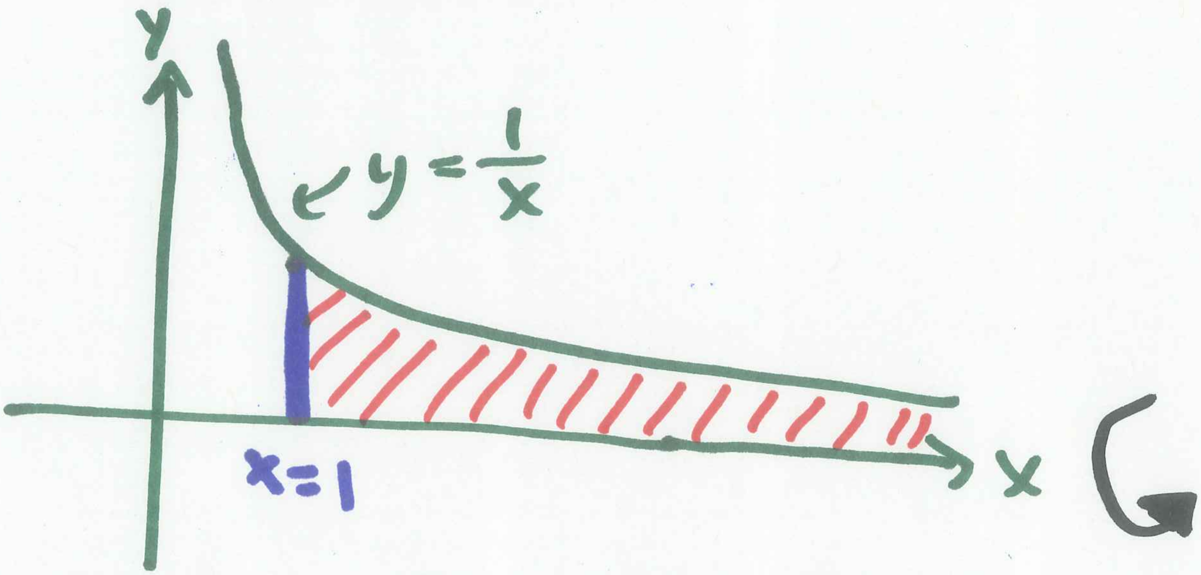
Ex (Gabriel's Horn):

Consider the solid of revolution given by rotating the region

Below $\frac{1}{x}$

Above x-axis

Right of $x=1$



What is its volume?

Disk Method

Radius: $\frac{1}{x}$

$$\int_1^\infty \pi \left(\frac{1}{x}\right)^2 dx = \lim_{b\to\infty} \int_1^b \pi x^{-2} dx$$

$$\int \pi x^{-2} dx = -\pi x^{-1} + C$$

$$= -\frac{\pi}{x} + C$$

$$\int_1^b \pi x^{-2} dx = -\pi b^{-1} - \left(-\pi (1)^{-1}\right)$$

$$= \pi - \pi/b$$

$$\lim_{b\to\infty} \pi - \frac{\pi}{b} = \pi - 0 \boxed{= \pi}$$

Ex (Gabriel's Horn): What is its surface area?

$$f(x) = \frac{1}{x} \qquad f'(x) = \frac{-1}{x^2}$$

$$\int_1^\infty 2\pi \cdot \frac{1}{x} \cdot \sqrt{1+\left(\frac{-1}{x^2}\right)^2}\,dx$$ DIVERGES

HARD

$$> \int_1^\infty 2\pi \frac{1}{x}\,dx$$ EASY

$+\infty$

$$\int 2\pi \frac{1}{x}\,dx = 2\pi \ln x + C$$

$$\lim_{b\to\infty}\left(\underline{2\pi \ln b} - 2\pi \ln 1\right) = \underline{\underline{\infty}}$$

DIVERGENT

Direct Comparison Test

If f & g are cts on $[a, \infty)$ and

$$0 \leq f(x) \leq g(x)$$

then

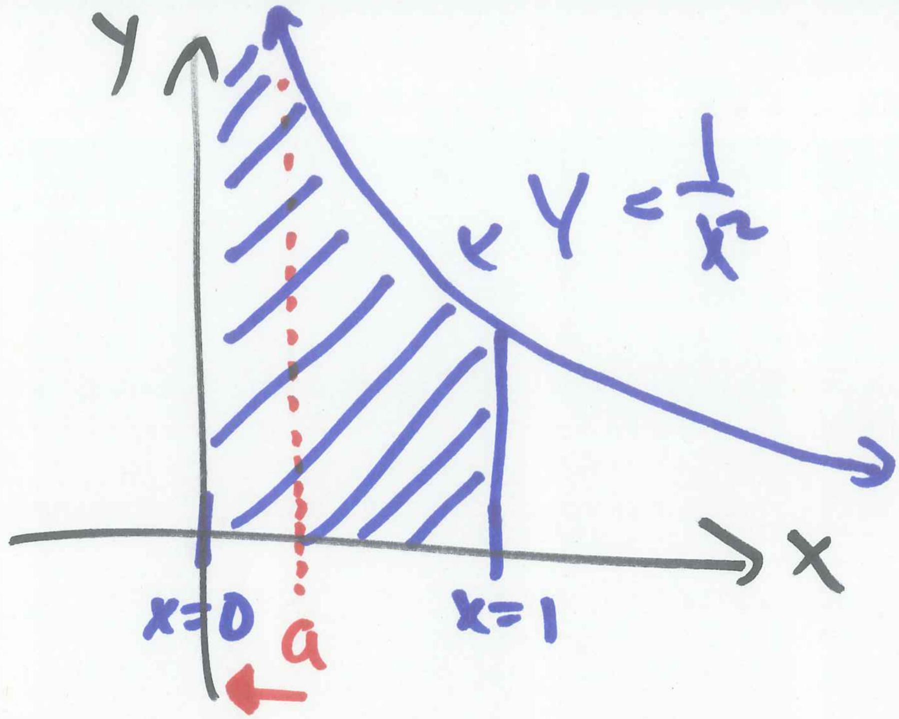
1. If $\int_a^\infty g(x)\,dx$ converges,
   then $\int_a^\infty f(x)\,dx$ converges.

   BIG to small

2. If $\int_a^\infty f(x)\,dx$ diverges
   then $\int_a^\infty g(x)\,dx$ diverges

   small to BIG

Ex: $\displaystyle\int_0^1 \frac{dx}{x^2}$



$$\downarrow$$

$$\lim_{a \to 0^+} \int_a^1 \frac{dx}{x^2}$$

. . . .

$$\int \frac{dx}{x^2} = \frac{-1}{x} + C$$

$$\int_a^1 \frac{dx}{x^2} = \frac{-1}{1} - \left(\frac{-1}{a}\right)$$

$$\lim_{a \to 0^+} \left(-1 + \frac{1}{a}\right) = -1 + (+\infty) = \underline{\underline{+\infty}}$$

Divergent

Improper Integrals: Type II

Is there a vertical asymptote?

1. If $f(x)$ is cts on $(a,b]$ and discont. at a, then

   $a<b$

$$\int_a^b f(x)\,dx = \lim_{c \to a^+} \int_c^b f(x)\,dx$$

2. If $f(x)$ is cts on $[a,b)$ and discontinuous at b, then

$$\int_a^b f(x)\,dx = \lim_{c \to b^-} \int_a^c f(x)\,dx$$

3. If $f(x)$ is disc. at c, $a<c<b$, then

$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$$

Converges iff BOTH converge!

Ex: $$\int_0^1 \frac{x+1}{\sqrt{x^2+2x}}\,dx$$

$$= \lim_{a\to 0^+} \int_a^1 \frac{x+1}{\sqrt{x^2+2x}}\,dx$$

$$\int \frac{x+1}{\sqrt{x^2+2x}}\,dx \quad = \frac{1}{2}\int \frac{du}{\sqrt{u}} = u^{1/2} + C$$

$$= \underline{\sqrt{x^2+2x} + C}$$

$u = x^2+2x$

$du = (2x+2)\,dx$

$$\int_a^1 \frac{x+1}{\sqrt{x^2+2x}}\,dx = \sqrt{1^2+2\cdot 1} - \sqrt{a^2+2a}$$

$$= \sqrt{3} - \sqrt{a^2+2a}$$

$$\lim_{a\to 0^+}\left(\sqrt{3} - \sqrt{a^2+2a}\right) = \underline{\sqrt{3}}$$

CONVERGES

Ex:

$$\int_0^2 \frac{dx}{\sqrt{4-x^2}}$$

Ex:

$$\int_1^\infty \frac{dx}{x\ln x}$$

LIMIT COMPARISON TEST

If f & g are cts on $[a, \infty)$ and

$$\lim_{x \to \infty} \frac{f(x)}{g(x)} = L, \quad 0 < L < \infty$$

then

$$\int_a^\infty f(x)\,dx \quad \text{and} \quad \int_a^\infty g(x)\,dx$$

both converge

— OR —

both diverge.

they behave the same

Ex: $\displaystyle\int_1^\infty \frac{dx}{x^2+x+5}$: Converges or Diverges?

$\displaystyle < \int_1^\infty \frac{dx}{x^2}$ ← Converges!

Converges by DCT.

Ex: $\cancel{\displaystyle\int_1^\infty \frac{2+5\sin^2 x}{x^2}\,dx}$

$$\int_1^\infty \frac{1+\frac{\sin x}{x}}{x^2}\,dx$$

Compare to $\frac{1}{x^2}$, but NOT DIRECTLY

$$f(x) = \frac{1 + \frac{\sin x}{x}}{x^2} \qquad g(x) = \frac{1}{x^2}$$

$$\lim_{x \to \infty} \frac{f(x)}{g(x)} = \lim_{x \to \infty} \frac{1 + \frac{\sin(x)}{x}}{\cancel{x^2}} \cdot \frac{\cancel{x^2}}{1}$$

$$= \lim_{x \to \infty} 1 + \frac{\sin(x)}{x}$$

$$= 1 + \lim_{x \to \infty} \frac{\sin(x)}{x}$$

$$= 1 = L$$

$$0 < L < \infty$$

$$\int_1^\infty \frac{1 + \frac{\sin x}{x}}{x^2}\, dx \qquad \int_1^\infty \frac{dx}{x^2}$$

BEHAVE THE SAME! Converges

Ex: Let p be a real number.

$$\int_a^\infty x^p \, dx$$