10.1 Sequences

Def: A **sequence** is a list of numbers

$$a_1, a_2, a_3, a_4, \dots, a_n, \dots$$

and these numbers are the **terms** of the sequence.

Ex: $\overset{a_1}{2}, \overset{a_2}{4}, \overset{a_3}{6}, \overset{a_4}{8}, \dots, \overset{a_n}{2 \cdot n}$

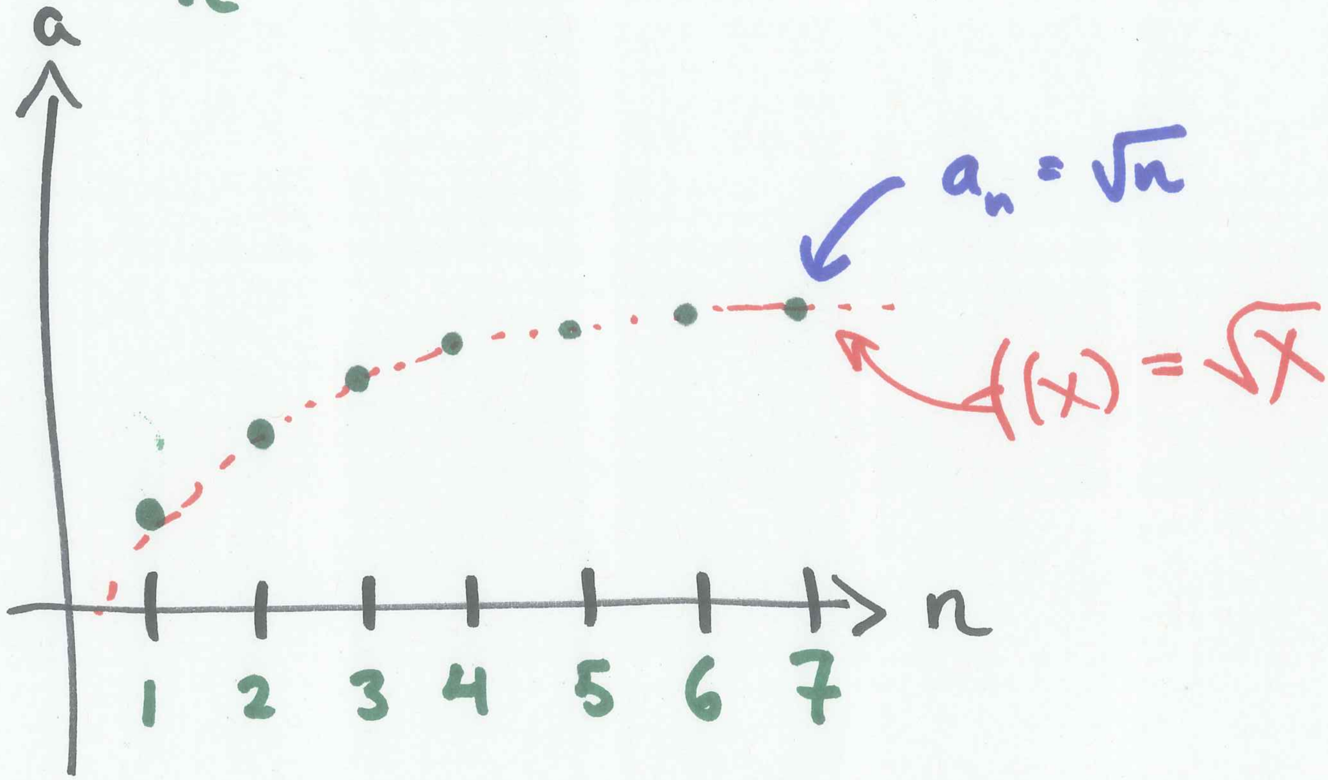
Ex: $\overset{a_1}{2}, \overset{a_2}{4}, \overset{a_3}{8}, \overset{a_4}{16}, \overset{a_5}{32} \leftarrow 2^5, \quad \overset{a_n}{2^n}$

Ex: $1, 1, 1, 1, \dots \overset{a_n}{1}$

Ex: $5, -1, 3567, 0,$

An infinite sequence is a function whose domain is the natural #'s.

$$a_n = a(n)$$



We can describe sequences in a few ways.

1. Defining terms

$$a_n = 2^n$$

2. Listing terms

$$\{a_n\} = \{2, 4, 8, 16, 32, \ldots\}$$

3. "Sequence notation"

$$\{a_n\} = \{2^n\}_{n=1}^{\infty}$$

Def: The sequence $\{a_n\}$ **converges** to the number L if for every $\varepsilon > 0$ there is an integer N such that

$$n > N \rightarrow |a_n - L| < \varepsilon.$$

If no such number L exists, then $\{a_n\}$ **diverges**.

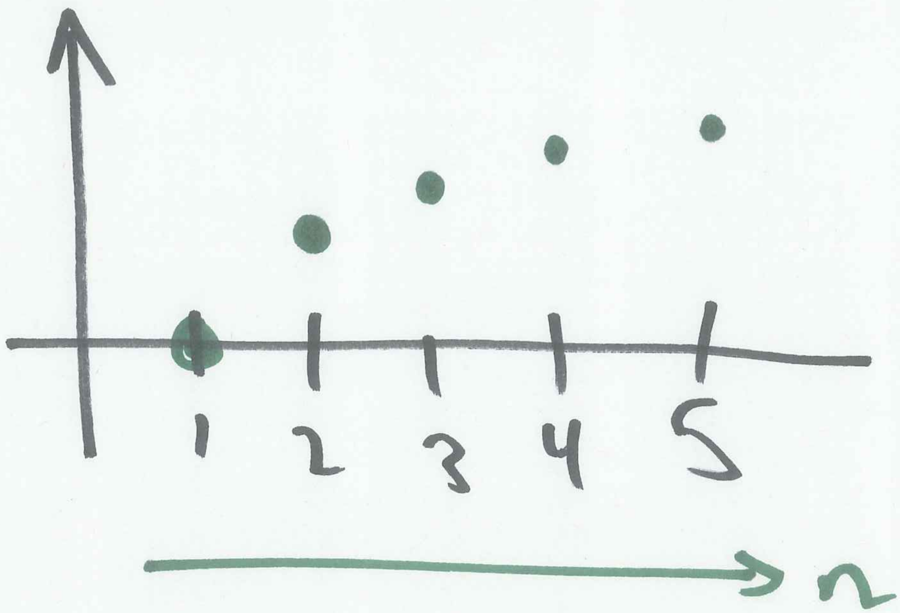
$$\lim_{n \to \infty} a_n = L \leftarrow \text{the } \underline{\text{limit}} \text{ of } a_n.$$

$$\boxed{a_n \to L}$$

(Recall Section 2.6)

Ex: $a_n = \dfrac{n-1}{n}$

$$\{a_n\} = \{0, \tfrac{1}{2}, \tfrac{2}{3}, \tfrac{3}{4}, \tfrac{4}{5}, \dots\}$$



What happens when

n is

REALLY BIG?

Def: Let $\{a_n\}$ be a sequence.

Diverges to Infinity:

If for all $M>0$ there is $N>0$ such that

$$n>N \rightarrow a_n > M.$$

Diverges to Negative Infinity:

If for all $M>0$ there is $N>0$ such that

$$n>N \rightarrow a_n < -M.$$

$$\lim_{n\to\infty} a_n = \infty \quad \text{-or-} \quad \lim_{n\to\infty} a_n = -\infty$$

THM 4: Suppose $f(x)$ is a function defined for all $x \geq n_0$ and that $\{a_n\}$ is a sequence with

$$a_n = f(n),$$

wherever $n \geq n_0$.

Then,

$$\lim_{n \to \infty} a_n = \lim_{x \to \infty} f(x).$$

(if $f(x)$ converges as $x \to \infty$)

Ex: $f(x) = \sin 2\pi x$

$a_n = f(n) = \sin 2\pi n = 0$

Ex: $a_n = \frac{\ln n}{n}$

$$f(x) = \frac{\ln x}{x} \quad (x \geq 1)$$

$$\lim_{x\to\infty} \frac{\ln x}{x} \underset{\text{L'Hopital!}}{=} \lim_{x\to\infty} \frac{1/x}{1} = \lim_{x\to\infty} \frac{1}{x}$$

$$= 0.$$

So $\lim_{x\to\infty} f(x) = 0$

$\longrightarrow \lim_{n\to\infty} a_n = 0.$

THM 1. Let $\{a_n\}$ and $\{b_n\}$ be sequences with

$$\lim_{n\to\infty} a_n = A, \quad \lim_{n\to\infty} b_n = B.$$

Then:

Sum Rule: $\lim_{n\to\infty} (a_n + b_n) = A + B$

Diff. Rule: $\lim_{n\to\infty} (a_n - b_n) = A - B$

Constant Mult Rule: $\lim_{n\to\infty} k \cdot a_n = k \cdot A$

Product Rule: $\lim_{n\to\infty} a_n \cdot b_n = A \cdot B$

Quotient Rule: $\lim_{n\to\infty} \frac{a_n}{b_n} = \frac{A}{B}$

(if $B \neq 0$)

THM 3 (CTS Function Theorem for Sequences):

Let $\{a_n\}$ be a sequence.

If $\lim_{n\to\infty} a_n = L$ and f is a function that is continuous at L, then

$$\lim_{n\to\infty} f(a_n) = f(L).$$

Ex: $b_n = \dfrac{\sqrt{n+5}}{\sqrt{n+1}} = \sqrt{\dfrac{n+5}{n+1}}$

$a_n = \dfrac{n+5}{n+1}$, $f(x) = \sqrt{x}$

$$\lim_{n\to\infty} \frac{n+5}{n+1} = \lim_{x\to\infty} \frac{x+5}{x+1} = \lim_{x\to\infty} \frac{1}{1} = \underline{\underline{1}}$$

$$\lim_{n\to\infty} \sqrt{\frac{n+5}{n+1}} = \sqrt{1} = 1.$$

THM 2 (Sandwich Theorem)

Let $\{a_n\}$, $\{b_n\}$, and $\{c_n\}$ be sequences with

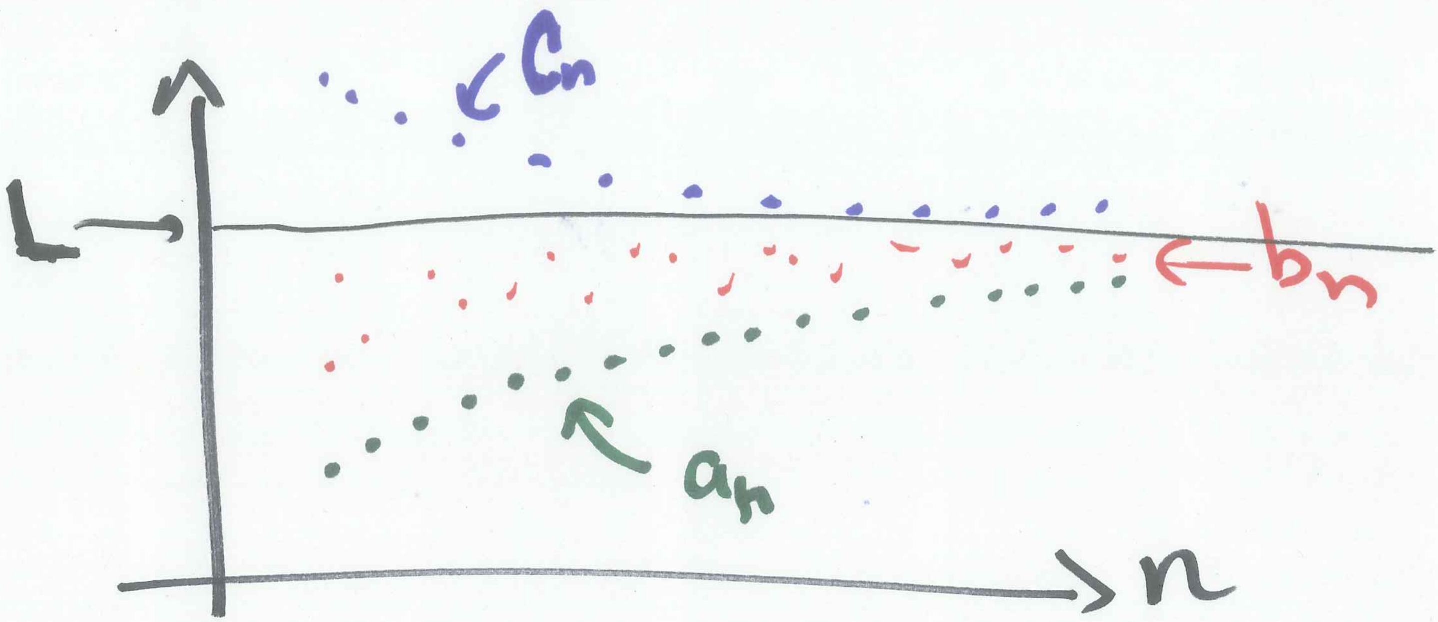
$$a_n \leq b_n \leq c_n.$$

If

$$\lim_{n\to\infty} a_n = L = \lim_{n\to\infty} c_n$$

Then

$$\lim_{n\to\infty} b_n = L.$$


c_n
L
b_n
a_n
n

Ex: $b_n = \dfrac{n\sin(n)}{3n^2+10}$

$$\underbrace{\frac{-n}{3n^2+10}}_{a_n} \leq \frac{n\sin(n)}{3n^2+10} \leq \underbrace{\frac{n}{3n^2+10}}_{c_n}$$

$$a_n \to 0 \qquad \frac{n\sin(n)}{3n^2+10} \to 0 \qquad c_n \to 0$$

by the

Sandwich Theorem

THM 5 (Common Limits)

$$\lim_{n\to\infty} \frac{\ln n}{n} = 0$$

$$\lim_{n\to\infty} x^{1/n} = 1 \quad (x>0)$$

$$\lim_{n\to\infty} \sqrt[n]{n} = 1$$

$$\lim_{n\to\infty} x^n = \begin{cases} 0 & \text{if } |x|<1 \\ 1 & \text{if } x=1 \\ \infty & \text{if } x>1 \\ \text{DNE} & \text{if } x\leq -1 \end{cases}$$

$$\lim_{n\to\infty} \left(1+\frac{x}{n}\right)^n = e^x$$

$$\lim_{n\to\infty} \frac{x^n}{n!} = 0$$

(x is constant w.r.t. n)

ORDERS OF GROWTH

$$1 \ll \ln n \ll n^p \ll x^n \ll n! \ll x^{n^2}$$

$(p>0)$ $(x>1)$

Ex: $$a_n = \frac{\cancel{1}+\cancel{\ln n}+\cancel{n^2}+e^n}{\cancel{n}+\cancel{e^n}+n!} \leftarrow \frac{e^n}{n!} \to 0$$