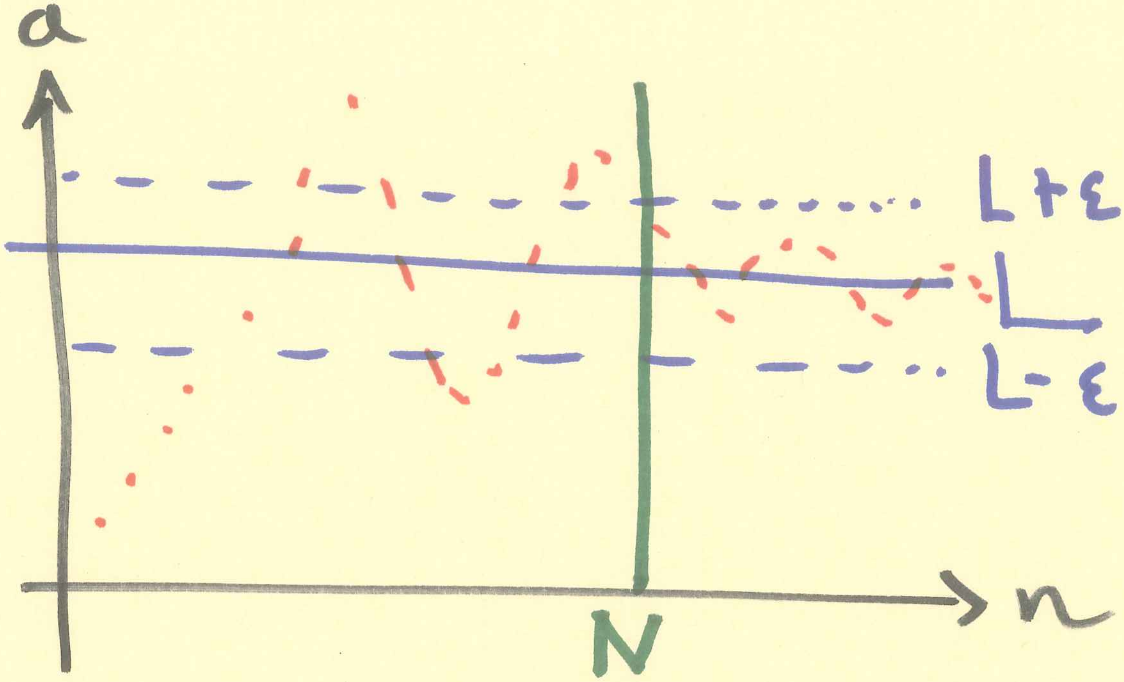
For all $\varepsilon > 0$ there is $N > 0$ such that

$$n > N \rightarrow |a_n - L| < \varepsilon.$$



Recursive Def. of Sequences

Ex: $F_0 = 0$, $F_1 = 1$,

$$F_{n+2} = F_n + F_{n+1}$$

$$\{F_n\} = \{0, 1, 1, 2, 3, 5, 8, 13, 21, \ldots\}$$

$$F_n = \left(\frac{1+\sqrt{5}}{2}\right)^n + \left(\frac{1-\sqrt{5}}{2}\right)^n$$

Ex: $a_0 = 1$, $a_{n+1} = 2 \cdot a_n$

$$\{a_n\} = \{1, 2, 4, 8, 16, 32, \ldots\}$$

$$a_n = 2^n \quad \left(a_{n+1} = 2 \cdot a_n = 2 \cdot 2^n = 2^{n+1}\right)$$

Def: For $\{a_n\}_{n=1}^{\infty}$ a sequence.

- a_n is nondecreasing if

$$a_n \leq a_{n+1}$$

- a_n is nonincreasing if

$$a_n \geq a_{n+1}$$

- a_n is monotonic if
  it is either nondecreasing
  or nonincreasing

$$b_0 = \frac{1}{6}, \quad b_1 = \frac{1}{3}$$

$$b_{n+2} = b_n^2 + b_{n+1}$$

$$\{b_n\} = \left\{ \frac{1}{6}, \frac{1}{3}, \frac{13}{36}, \frac{17}{36}, \frac{781}{1296}, \dots \right.$$

$$C_{n+2} = C_n + C_{n+1}$$

$$\frac{F_{n+1}}{F_n} = \frac{F_n + F_{n-1}}{F_n} = 1 + \frac{F_{n-1}}{F_n}$$

$$x_n = 1 + \frac{1}{x_{n-1}}$$

Def: A sequence is bounded from above if $a_n \leq M$ (always) for some M.

A sequence is bounded from below if $a_n \geq M$ (always) for some M.

For $\{a_n\}_{n=1}^{\infty}$,

least upper bound:

<u>minimum</u> M s/t $a_n \leq M$ always

greatest lower bound:

maximum M s/t $a_n \geq M$ always

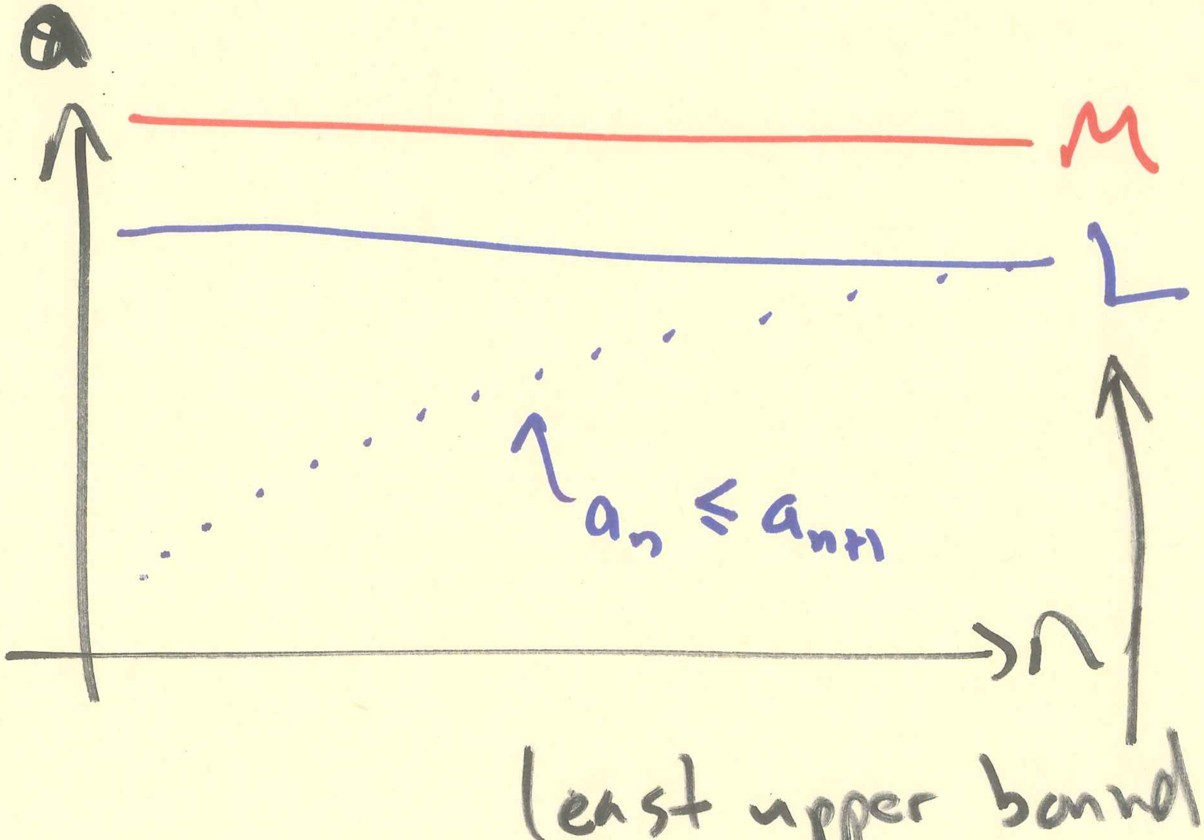
bounded / unbounded

Thm 6 (Monotonic Sequence Thm)

If $\{a_n\}$ is monotonic and bounded, then
(from above and below)

$\lim_{n\to\infty} a_n$ exists.

(the sequence converges)



10.2: Infinite Series

Let $\{a_n\}_{n=1}^{\infty}$ be a sequence. The infinite series generated by a_n is

$$\sum_{n=1}^{\infty} a_n = \lim_{N\to\infty} \underbrace{\sum_{n=1}^{N} a_n}_{S_N}$$

N^{th} partial sum.

$$S_0 = 0, \quad S_{N+1} = S_N + a_{N+1}$$

Ex: A geometric series

is a sum

$$\sum_{i=0}^{\infty} a \cdot r^i = a + ar + ar^2 + \cdots$$

$$\sum_{i=0}^{n-1} a \cdot r^i = \frac{a(1-r^n)}{1-r} \qquad (|r|<1) \quad (-1<r<1)$$

$$\downarrow$$

$$\frac{a}{1-r} = \sum_{i=0}^{\infty} a \cdot r^i$$

Ex: $\sum_{i=0}^{\infty} \left(\frac{1}{2}\right)^i = 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots$

$a = 1$

$r = \frac{1}{2}$, $\frac{a}{1-r} = \frac{1}{1-\frac{1}{2}} = \underline{\underline{2}}$

$1, \quad \frac{3}{2}, \quad \frac{7}{4}, \quad \frac{15}{8} \quad \cdots$

Thm 7: If $\sum_{n=1}^{\infty} a_n$ converges,

then $a_n \to 0$.

The n^{th} Term Test

Consider $\sum_{n=1}^{\infty} a_n$.

- If $a_n \not\to 0$, then $\sum_{n=1}^{\infty} a_n$ DIVERGES

- If $a_n \to 0$, then this test is INCONCLUSIVE

Ex: $$\sum_{n=1}^{\infty}\left(\frac{1}{n(n+1)}\right) = \sum_{n=1}^{\infty}\left(\frac{1}{n}-\frac{1}{n+1}\right)$$

$$= \frac{1}{1} - \cancel{\frac{1}{2}}$$

$$\cancel{+\frac{1}{2}} - \cancel{\frac{1}{3}}$$

$$\cancel{+\frac{1}{3}} - \frac{1}{4}$$

$$\vdots$$

$$\sum_{n=1}^{N}\left(\frac{1}{n}-\frac{1}{n+1}\right) = \left(\frac{1}{1}-\frac{1}{2}\right)+\left(\frac{1}{2}-\frac{1}{3}\right)+\cdots$$

$$+\left(\frac{1}{N}-\frac{1}{N+1}\right)$$

$$= 1 - \frac{1}{N+1} \longrightarrow 1$$

Bad Ex: $\sum_{n=1}^{\infty}(2^n - 2^{n+1})$ ← Diverges!

Ex!

- $\sum_{i=1}^{n} \left(1+\frac{1}{n}\right)^{n}$

  $(1+\frac{1}{n})^n \to e \neq 0$

  DIVERGES by n^{th} term test.

- $\sum_{n=1}^{\infty} \frac{1}{n}$

  $\frac{1}{n} \to 0$

  n^{th} term test is INCONCLUSIVE

- $\sum_{n=1}^{\infty} \frac{1}{n^2}$

  $\frac{1}{n^2} \to 0$

  INCONCLUSIVE

- $\sum_{n=1}^{\infty} \sin n$

  $\lim_{n\to\infty} \sin n$ DNE

  $\sin n \not\to 0$

  DIVERGES

  by n^{th} term test

Thm 8. If $\sum_{n=1}^{\infty} a_n = A$ and

$\sum_{n=1}^{\infty} b_n = B$, then

- $\sum_{n=1}^{\infty} (a_n + b_n) = A + B$

- $\sum_{n=1}^{\infty} (a_n - b_n) = A - B$

- $\sum_{n=1}^{\infty} k a_n = k \cdot A$

(k a constant.)

Ex: $\sum_{n=1}^{\infty} \left(\frac{1}{2^n} + \frac{1}{3^n} + \frac{1}{n(n+1)} \right)$