10.3 The Integral Test

Idea: Use improper integrals to understand infinite series.

$$\int_1^\infty f(x)\,dx \longleftrightarrow \sum_{n=1}^{\infty} f(n)$$

Do They Behave the Same?

$$\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2}$$

$$\left. + \frac{1}{3} + \frac{1}{4} \right\} \geq \frac{1}{2}$$

$$\left. + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} \right\} \geq \frac{1}{2}$$

$$\left. + \frac{1}{9} + \frac{1}{10} + \cdots + \frac{1}{16} \right\} \geq \frac{1}{2}$$

$$\left. + \frac{1}{17} + \cdots + \frac{1}{32} \right\} \geq \frac{1}{2}$$

$$\left. + \frac{1}{33} + \cdots + \frac{1}{64} \right\} \geq \frac{1}{2}$$

$$+ \cdots$$

$$\frac{1}{2^p}$$

$$\sum_{n=1}^{N = 2^p} \frac{1}{n} \geq p \cdot \frac{1}{2} = \frac{1}{2} \log_2 N$$

Note: If $a_n \geq 0$ always, then

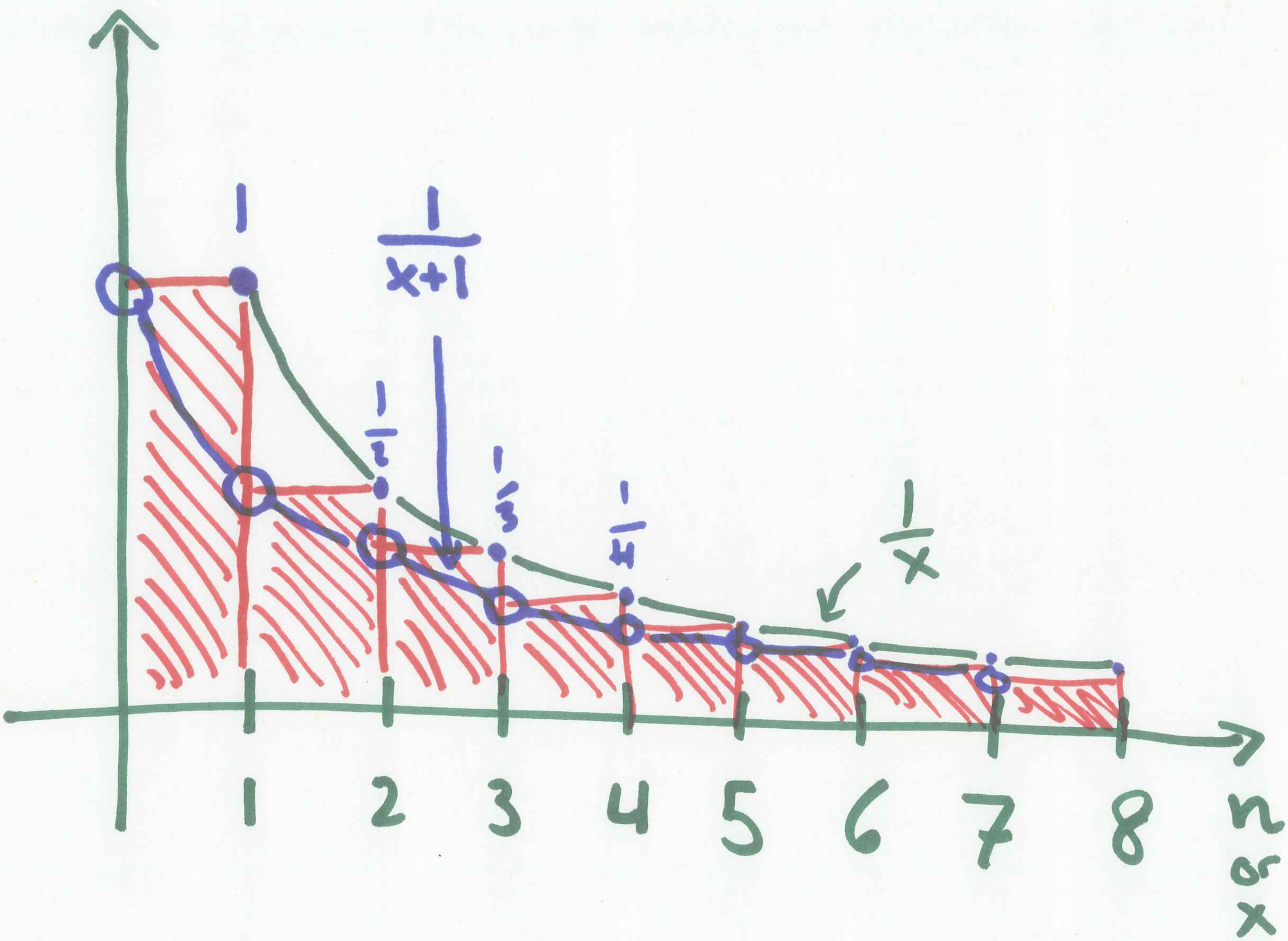
$$s_N = \sum_{n=1}^{N} a_n$$

is <u>nondecreasing</u> (monotonic) and converges if and only if

$$s_N \leq M \text{ always}$$

for some M.

$$\sum_{n=1}^{\infty} \frac{1}{n} = \frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \cdots$$



Thm 9 (The Integral Test)

Let $\{a_n\}$ be a sequence of <u>positive</u> terms.

Suppose $a_n = f(n)$ for a function $f(x)$ that is

- continuous
- positive
- decreasing

over $[N, \infty)$ for some N.

Then

$$\int_N^\infty f(x)\,dx \quad \text{and} \quad \sum_{n=N}^\infty f(n) = \sum_{n=N}^\infty a_n$$

both diverge or both converge.

Ex: Does the series $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converge or diverge?

$f(x) = \frac{1}{x^2}$, cts on $[1, \infty)$
decreasing
positive!

$$\int_1^{\infty} \frac{1}{x^2}\,dx = \lim_{b\to\infty} \int_1^b \frac{dx}{x^2}$$

$$= \lim_{b\to\infty} \left[-x^{-1}\right]_1^b$$

$$= \lim_{b\to\infty} \left[\frac{-1}{b} + \frac{1}{1}\right] = \underline{\underline{1}}$$

$\int_1^{\infty} \frac{dx}{x^2}$ converges, so

$\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges by the Integral Test.

Ex: Does the series $\sum_{n=2}^{\infty} \frac{1}{n \ln n}$ converge or diverge?

$$f(x) = \frac{1}{x \ln x}$$

$$\int_2^{\infty} \frac{dx}{x \ln x} = \lim_{b \to \infty} \int_2^{b} \frac{dx}{x \ln x}$$

$$\left[\int \frac{dx}{x \ln x} = \int \frac{du}{u} = \ln|u| + C \right.$$

$u = \ln x$

$du = \frac{dx}{x}$

$$\left. = \ln \ln x + C \right]$$

$$= \lim_{b \to \infty} \left[\ln \ln b - \ln \ln 2 \right]$$

$$= \infty$$

$\int_2^{\infty} \frac{dx}{x \ln x}$ diverges, so $\sum_{n=2}^{\infty} \frac{1}{n \ln n}$ diverges by I.T.

Ex: Does the series $\displaystyle\sum_{n=2}^{\infty} \frac{1}{\sqrt{n^2-1}}$

converge or diverge?

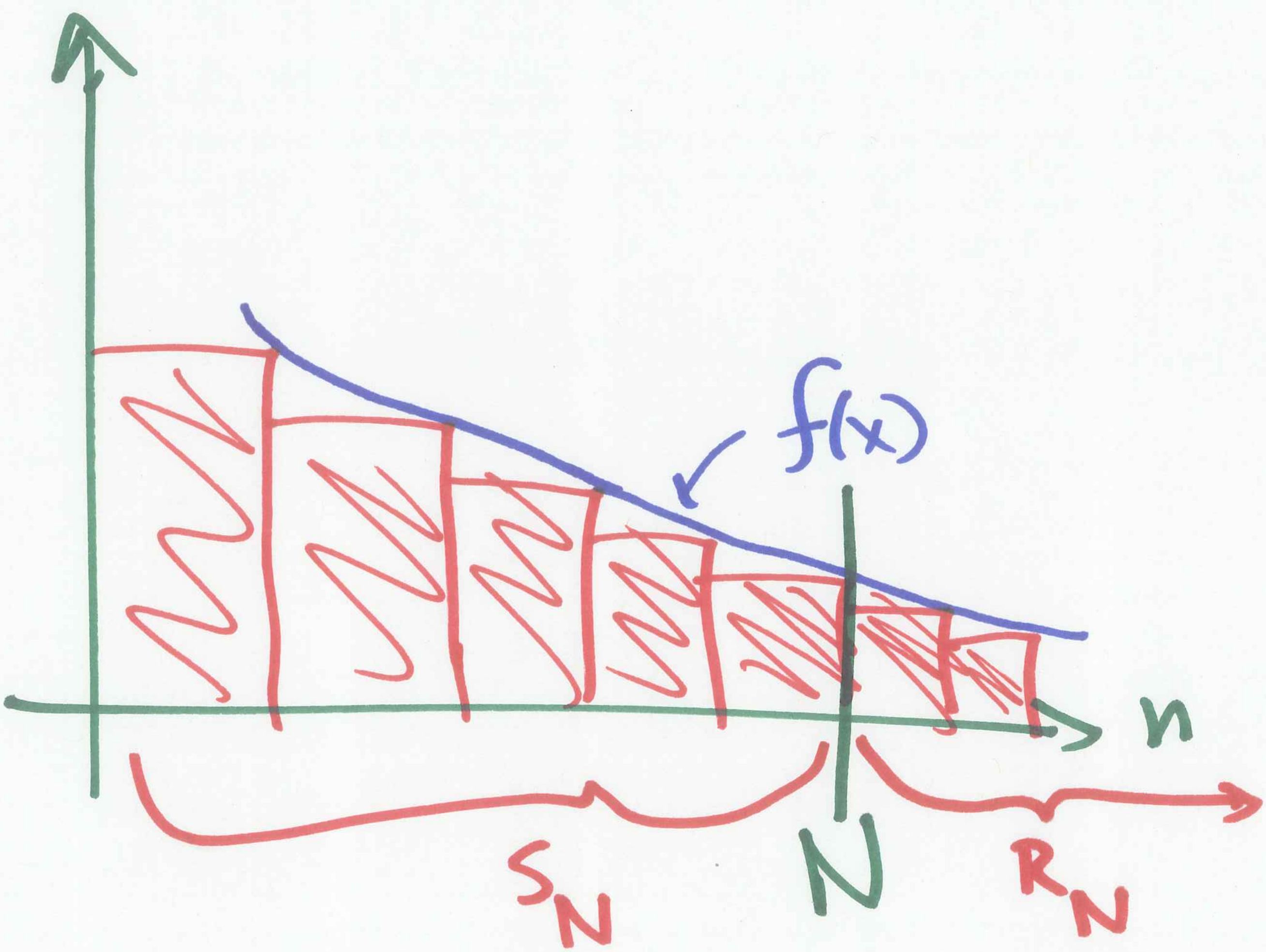
Behaves like $\dfrac{1}{n}$

$$\sum_{n=1}^{\infty} \frac{1}{n^p} \begin{cases} \text{Converge if } p>1 \\ \text{Diverge if } p\leq 1 \end{cases}$$

Error Estimation:

Suppose $\sum_{n=1}^{\infty} a_n$ converges.

Can we determine its value?



Bounds for Remainder by Integral Test

Let $a_n = f(n)$ where

$f(x)$ is continuous, positive, decreasing

$$S_N = \sum_{n=1}^{N} a_n, \quad R_N = \left| S_N - \sum_{n=1}^{\infty} a_n \right|$$

Remainder

Converges

$$\int_{N+1}^{\infty} f(x)\,dx \leq R_N \leq \int_{N}^{\infty} f(x)\,dx$$

(picture)

Ex: How many terms are required to determine the value of $\sum_{n=1}^{\infty} \frac{1}{n^2}$ up to an error of at most 10^{-3}? Want: $R_N \leq 10^{-3}$

$$R_N \leq \int_N^{\infty} \frac{dx}{x^2} = \lim_{b\to\infty}\left[\frac{-1}{b} + \frac{1}{N}\right]$$

$$= \frac{1}{N} \leq 10^{-3}$$

$$N \geq 10^3 = 1{,}000.$$