10.4 Comparison Tests

Thm 10: (Direct Comparison Test)
Let $\sum a_n$ $\sum c_n$ and $\sum d_n$ be series with

$$0 \leq c_n \leq a_n \leq d_n.$$

a) If $\sum c_n$ diverges, then so does $\sum a_n$

b) If $\sum d_n$ converges, then so does $\sum a_n$

Ex: $\sum_{n=1}^{\infty} \frac{1}{10n-2}$

"Important": $\sum_{n=1}^{\infty} \frac{1}{10n}$ ← Diverges by Harmonic Series.

$$0 \leq \underbrace{\frac{1}{10n}}_{c_n} \leq \underbrace{\frac{1}{10n-2}}_{a_n}$$

Thus, $\sum_{n=1}^{\infty} \frac{1}{10n-2}$ <u>diverges</u>

by DCT

Ex: $\sum_{n=1}^{\infty} \frac{1}{n^2+5n+1}$

Consider $\sum_{n=1}^{\infty} \frac{1}{n^2}$ ← Converges by Integral Test (p-Test)

$$0 \leq \frac{1}{n^2+5n+1} \leq \frac{1}{n^2}$$

Thus, $\sum \frac{1}{n^2+5n+1}$ Converges by DCT.

Thm 11: (Limit Comparison Test)

Suppose that $a_n > 0$ and $b_n > 0$ for all $n \geq N$ (for some N)

1. If $\lim_{n\to\infty} \frac{a_n}{b_n} = c > 0$, then $\sum a_n$ & $\sum b_n$ either both converge or both diverge.

2. If $\lim_{n\to\infty} \frac{a_n}{b_n} = 0$ and $\sum b_n$ converges, then $\sum a_n$ converges

3. If $\lim_{n\to\infty} \frac{a_n}{b_n} = \infty$ and $\sum b_n$ diverges, then $\sum a_n$ diverges.

Ex: $\displaystyle\sum_{n=5}^{\infty} \frac{1}{n^2-5n+1}$

Considers $\sum \frac{1}{n^2}$ ← Converges

$$\underbrace{\frac{1}{n^2}}_{b_n} \leq \underbrace{\frac{1}{n^2-5n+1}}_{a_n}$$

Smaller!

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} a_n \cdot b_n^{-1} =$$

$$= \lim_{n\to\infty} \frac{1}{n^2-5n+1} \cdot \frac{n^2}{1}$$

$$= \lim_{n\to\infty} \frac{1}{1-5/n+1/n^2} = 1 > 0$$

Ex: $\sum_{n=1}^{\infty} \frac{\ln n}{n^{3/2}}$ $\qquad \sum \frac{1}{n^{3/2}}$

$\uparrow$ a_n $\qquad \uparrow$ b_n

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{\ln n}{\cancel{n^{3/2}}} \cdot \frac{\cancel{n^{3/2}}}{1} = \lim_{n\to\infty} \ln n = \infty$$

$$c_n = \frac{n^{1/4}}{n^{3/2}} = \frac{1}{n^{5/4}}$$ ($\sum \frac{1}{n^{5/4}}$ converges by p-test)

$$\lim_{n\to\infty} \frac{a_n}{c_n} = \lim_{n\to\infty} \frac{\ln n}{n^{3/2}} \cdot \frac{n^{5/4}}{1}$$

$$= \lim_{n\to\infty} \frac{\ln n}{n^{1/4}}$$

L'Hopital

$$= \lim_{n\to\infty} \frac{1/n}{\frac{1}{4} n^{-3/4}}$$

$$= \lim_{n\to\infty} \frac{4}{n^{1/4}} = 0.$$

$\boxed{\sum a_n \text{ converges by LCT}}$

Ex: $$\sum_{n=1}^{\infty} \frac{n^2+n+1}{n^3+5n^2-n+10}$$

$$\frac{n^2}{n^3} = \frac{1}{n}.$$

$\sum_{n=1}^{\infty} \frac{1}{n}$ diverges by p-test.

$$\lim_{n\to\infty} \frac{n^2+n+1}{n^3+5n^2-n+10} \cdot \frac{n}{1}$$

$$= \lim_{n\to\infty} \frac{n^3+n^2+n}{n^3+5n^2-n+10}$$

$$= \lim_{n\to\infty} \frac{1+\frac{1}{n}+\frac{1}{n^2}}{1+\frac{5}{n}-\frac{1}{n^2}+\frac{10}{n^3}} = 1 > 0$$

Thus, $\sum \frac{n^2+n+1}{n^3+5n^2-n+10}$ diverges by LCT.