# 10.5 The Ratio & Root Tests

Recall Geometric Series:

$$\sum_{n=0}^{\infty} a \cdot r^n = \frac{a}{1-r} \quad \left(\text{if } |r|<1\right)$$

Let

$\sum_{n=0}^{\infty} b_n$ be a complicated series.

"Does it **LOOK** Geometric?"

# Thm 12: The Ratio Test

Let $\sum a_n$ be a series with $a_n > 0$ and suppose that

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \rho$$

a) $\sum a_n$ converges if $\underline{\rho < 1}$

b) $\sum a_n$ diverges if $\underline{\rho > 1}$

c) test is <u>inconclusive</u> if $\underline{\rho = 1}$

Ex: $\displaystyle\sum_{n=1}^{\infty} \frac{n}{3^n}$ ← Converges by Ratio Test

$$\lim_{n\to\infty} \frac{(n+1)/3^{n+1}}{n/3^n} = \lim_{n\to\infty} \frac{n+1}{n} \cdot \frac{3^n}{3^{n+1}} = \frac{1}{3} < 1$$

Ex: $\displaystyle\sum_{n=1}^{\infty} \frac{n!}{2^n}$ ← Diverges by Ratio Test

$$\lim_{n\to\infty} \frac{(n+1)!/2^{n+1}}{n!/2^n} = \lim_{n\to\infty} \frac{\overbrace{(n+1)!}^{(n+1)(\cancel{n!})}}{\cancel{n!}} \cdot \frac{2^{\cancel{n}}}{2^{\cancel{n+1}}} = \lim_{n\to\infty} \frac{n+1}{2} = \infty$$

Ex: $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2}$ ← Ratio Test is <u>inconclusive</u>

$$\lim_{n\to\infty} \frac{1/(n+1)^2}{1/n^2} = \lim_{n\to\infty} \frac{n^2}{(n+1)^2} = \lim_{n\to\infty} \frac{n^2}{n^2+2n+1}$$

$$= \lim_{n\to\infty} \frac{1}{1+\frac{2}{n}+\frac{1}{n^2}} = 1$$

# Thm 13: The Root Test

Let $\sum a_n$ be a series with $a_n \geq 0$ for $n \geq N$ and suppose

$$\lim_{n\to\infty} \sqrt[n]{a_n} = \rho$$

a) $\sum a_n$ converges if $\rho < 1$

b) $\sum a_n$ diverges if $\rho > 1$

c) test is inconclusive if $\rho = 1$

Ex: $\sum_{n=1}^{\infty} \left(\frac{3}{4}\right)^n \cdot n^2$ ← Converges by Root Test

$$\lim_{n\to\infty} \sqrt[n]{\left(\frac{3}{4}\right)^n \cdot n^2} = \lim_{n\to\infty} \underbrace{\sqrt[n]{\left(\frac{3}{4}\right)^n}}_{\frac{3}{4}} \cdot \underbrace{\sqrt[n]{n^2}}_{\left(\sqrt[n]{n}\right)^2} = \frac{3}{4} < 1$$

← goes to 1

Ex: $\sum_{n=1}^{\infty} \frac{3^n}{n^n} = \sum_{n=1}^{\infty} \left(\frac{3}{n}\right)^n$ ← Converges by Root Test

$$\lim_{n\to\infty} \sqrt[n]{\left(\frac{3}{n}\right)^n} = \lim_{n\to\infty} \frac{3}{n} = 0 < 1$$

Ex: $\sum_{n=1}^{\infty} \frac{n!}{n^n}$ ← Compare to

$$\sum_{n=1}^{\infty} \frac{(n)^{n/2} \left(\frac{n}{2}\right)^{n/2}}{n^n}$$

# Recursive Definitions

$a_1 = 2 \qquad a_{n+1} = \frac{1+\sin n}{n} a_n$

Does $\sum a_n$ converge or diverge?

Ratio Test:

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{\frac{1+\sin n}{n} \cdot \cancel{a_n}}{\cancel{a_n}}$$

$$= \lim_{n\to\infty} \frac{1+\sin n}{n}$$

$$= 0 < 1$$

$\sum a_n$ converges by Ratio Test.

Ex: $\sum_{n=1}^{\infty}\left(\frac{1}{n}-\frac{1}{n^2}\right)^n$

Root Test:

$$\lim_{n\to\infty}\sqrt[n]{\left(\frac{1}{n}-\frac{1}{n^2}\right)^n} = \lim_{n\to\infty}\frac{1}{n}-\frac{1}{n^2}$$

$$= 0 < 1.$$

Ex: $$\sum_{n=1}^{\infty} \frac{2^n\, n!\, n!}{(2n)!}$$

__Ratio Test:__

$$\lim_{n\to\infty} \frac{2^{\cancel{n}+1}\,\overset{(n+1)\cdot\cancel{n!}}{(n+1)!}\,\overset{(n+1)\cdot\cancel{n!}}{(n+1)!}}{\underbrace{(2(n+1))!}_{\underbrace{(2n+2)!}_{(2n+2)\cdot(2n+1)\cdot\cancel{(2n)!}}}} \cdot \frac{\cancel{(2n)!}}{\cancel{2^n}\,\cancel{n!}\,\cancel{n!}}$$

$$= \lim_{n\to\infty} \frac{2(n+1)(n+1)}{(2n+2)(2n+1)}$$

$$= \lim_{n\to\infty} \frac{2n^2+4n+2}{4n^2+6n+2}$$

$$= \lim_{n\to\infty} \frac{1+\frac{2}{n}+\frac{1}{n^2}}{2+\frac{3}{n}+\frac{1}{n^2}} = \frac{1}{2} < 1$$

Ex: $\displaystyle\sum_{n=1}^{\infty} \frac{n^n}{(2^n)^2} = \sum_{n=1}^{\infty} \frac{n^n}{2^{2n}} = \sum \frac{n^n}{4^n}$

$\displaystyle\lim_{n\to\infty} \sqrt[n]{\frac{n^n}{2^{2n}}} = \lim_{n\to\infty} \frac{n}{2^2} = \infty > 1$

Ex: $\displaystyle\sum_{n=1}^{\infty} \frac{n^n}{2^{(n^2)}}$

$$\lim_{n\to\infty} \sqrt[n]{\frac{n^n}{2^{(n^2)}}} = \lim_{n\to\infty} \frac{n}{2^{n^2\cdot\frac{1}{n}}}$$

$$= \lim_{n\to\infty} \frac{n}{2^n} = 0 < 1$$