Series Tests for Convergence/Divergence

Common Series
- $\sum \frac{1}{n^p}$ (p-Test)
- $\sum a \cdot r^n$ (Geometric)
- (Telescoping)

nth-Term Test

Integral Test

Direct/Limit Comparison Test

Ratio Test

Root Test

nonnegative sequences only

p-Series

$\sum \frac{1}{n^p}$ converges iff $p > 1$.

diverges iff $p \leq 1$

Ex: $\sum \frac{1}{n}$ (Harmonic)

(prove by Int. Test)

$\sum \frac{1}{n^2}$

$\sum \frac{1}{\sqrt{n}}$

$\sum \frac{1}{n^3}$

$\sum \left(\frac{1}{\sqrt{n}}\right)^3$

Geometric Series

$$\sum_{n=0}^{\infty} a \cdot r^n = \frac{a}{1-r} \text{ iff } |r|<1$$

Diverges iff $|r| \geq 1$

Ex: $$\sum_{n=0}^{\infty} \frac{1}{2^n} = \frac{1}{1-\frac{1}{2}} = \frac{1}{1/2} = 2$$

$$\sum_{n=0}^{\infty} 3 \cdot \left(\frac{2}{3}\right)^n$$

$$\sum_{n=0}^{\infty} \frac{3^n}{2^{2n+1}} = \sum_{n=0}^{\infty} \frac{3^n}{2 \cdot (2^2)^n} = \sum_{n=0}^{\infty} \frac{1}{2} \cdot \left(\frac{3}{4}\right)^n$$

n=0 term

$$* \quad \sum_{n=1}^{\infty} \left(\frac{1}{3}\right)^n = \left(\sum_{n=0}^{\infty} \left(\frac{1}{3}\right)^n\right) - \left(\frac{1}{3}\right)^0$$

$$= \frac{1}{1-\frac{1}{3}} - 1 = \underline{\frac{1}{2}}$$

Telescoping Sums (rare)

$$\sum_{n=0}^{\infty} (a_{n+1} - a_n) = -a_0 \text{ iff } a_n \to 0$$

Ex: $$\sum_{n=1}^{\infty} \left(\frac{1}{n+1} - \frac{1}{n} \right)$$

$$\sum_{n=1}^{\infty} \left(\frac{3}{4^n} - \frac{3}{4^{n-1}} \right)$$

* $$\sum_{n=0}^{\infty} \frac{1}{n^2+3n+2}$$

n^{th}-term Test

What: If $a_n \not\to 0$, then $\sum a_n$ <u>diverges</u>

When: <u>Always</u> check (mentally) if $a_n \to 0$.

Ex: $$\sum_{n=1}^{\infty} \left(1 - \frac{x}{n}\right)^n$$

$$\sum_{n=1}^{\infty} \sin n$$

$$\sum_{n=1}^{\infty} \left(\frac{1001}{1000}\right)^n$$

Integral Test

What: $\sum f(n)$ and $\int_N^\infty f(x)\,dx$ behave the same!

When:

- $f(x)$ is cts, positive, decreasing
- $\int f(x)\,dx$ is not terrible.

Ex:

$$\sum_{n=1}^{\infty} \frac{1}{n}$$

$$\sum_{n=2}^{\infty} \frac{1}{n \ln n} \longleftrightarrow \int_2^\infty \frac{dx}{x(\ln x)^2} \quad \text{DIVERGES}$$

$$\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^2} \longleftrightarrow \int_2^\infty \frac{dx}{x(\ln x)^2} \quad \text{CONVERGES}$$

$$\sum_{n=0}^{\infty} \frac{1}{e^n}$$

BONUS: We can use $\int_N^\infty f(x)\,dx$ for error estimation!

~~Direct~~/Limit Comparison Test

What:

- Find $\sum b_n$ where b_n is "like" a_n.
- Compute $\lim_{n\to\infty} \frac{a_n}{b_n} = L$
- If $0 < L < \infty$, then
  $\sum a_n$ & $\sum b_n$ behave the <u>same</u>
- If $L = \infty$ then
  $\sum b_n$ diverges $\to$ $\sum a_n$ diverges
- If $L = 0$, then
  $\sum b_n$ converges $\to$ $\sum a_n$ converges

~~Ex:~~ **When:**

- If a_n is a ratio between two polynomials.
- If mixing polynomials and ln's — replace with n^{ε} small $\varepsilon > 0$.
- Summations in a fraction

Limit Comparison Test

Ex: $\displaystyle\sum_{n=1}^{\infty} \frac{n^2+3n+1}{n^3+2n-1} \xrightarrow[\text{to}]{\text{Compare}} \sum \frac{n^2}{n^3} = \underline{\sum \frac{1}{n}}$

Diverges

$\displaystyle\sum_{n=1}^{\infty} \frac{\ln n}{n^2} \xrightarrow{\text{Replace}} n^{1/2}$

$\xrightarrow[\text{Compare to}]{} \displaystyle\sum \frac{n^{1/2}}{n^2} = \sum \frac{1}{n^{3/2}}$

Converges

$\displaystyle * \sum_{n=1}^{\infty} \frac{e^{n^2+n+1}}{\pi^{3n^2-n+1000}} \xrightarrow[\text{to}]{\text{Compare}} \underbrace{\sum \frac{e^{n^2}}{\pi^{3n^2}}}_{\text{Use Root TEST!}}$

$\displaystyle\sum_{n=1}^{\infty} \frac{n^3+(\sin n)\cdot n^2}{n^4+(\cos n)\, n^3}$

Ratio Test

What:
- Compute $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = L$
- If $L<1$, then $\sum a_n$ converges
- If $L>1$, then $\sum a_n$ diverges

When:
- "Looks geometric"
- Uses exponentials & factorials

Ex: $\sum_{n=1}^{\infty} \frac{e^n}{n!}$ $\quad \lim \frac{e^{n+1}}{(n+1)!} \cdot \frac{n!}{e^n} = \lim \frac{e}{n+1} = 0$

$\sum_{n=1}^{\infty} \frac{n! \cdot n!}{(2n)!}$

* $\sum_{n=1}^{\infty} \frac{n \cdot 2^n}{n^2 \cdot 3^n}$

gets complicated when writing a_{n+1}

(Try Root Test First)

Root Test

What:
- Compute $\lim_{n\to\infty} \sqrt[n]{a_n} = L$
- If $L<1$, then $\sum a_n$ converges
- If $L>1$, then $\sum a_n$ diverges

When:
- If $\lim_{n\to\infty} \sqrt[n]{a_n}$ is easy to compute
- Uses exponential functions, but <u>not</u> factorials.
- Has complicated polynomials as part of exponential func.

Note: If $p(n)$ is polynomial, then $\lim \sqrt[n]{p(n)} = 1$

Ex:

$$\sum_{n=1}^{\infty} \left(\frac{1}{n}-\frac{1}{n^2}\right)^n \qquad \sum_{n=1}^{\infty} \frac{n2^n}{n^2 3^n}$$

$$\sum_{n=1}^{\infty} \frac{\pi^n}{e^{n^2}} \qquad \sum_{n=1}^{\infty} (n^3+n+1000)\left(\frac{1}{2}\right)^n$$