10.6 Alternating Series, Absolute and Conditional Convergence

Def: A series whose terms alternate between positive and negative terms are <u>alternating series</u>.

Ex:

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} = \frac{1}{1} - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} \cdots$$

$$\sum_{n=0}^{\infty} \left(\frac{-1}{2}\right)^n = 1 - \frac{1}{2} + \frac{1}{4} - \frac{1}{8} + \frac{1}{16} \cdots$$

$$\sum_{n=0}^{\infty} (-1)^n \frac{e^n}{n!} = 1 - e + \frac{e^2}{2} - \frac{e^3}{6} + \frac{e^4}{24} \cdots$$

Thm 14 (The Alternating Series Test)

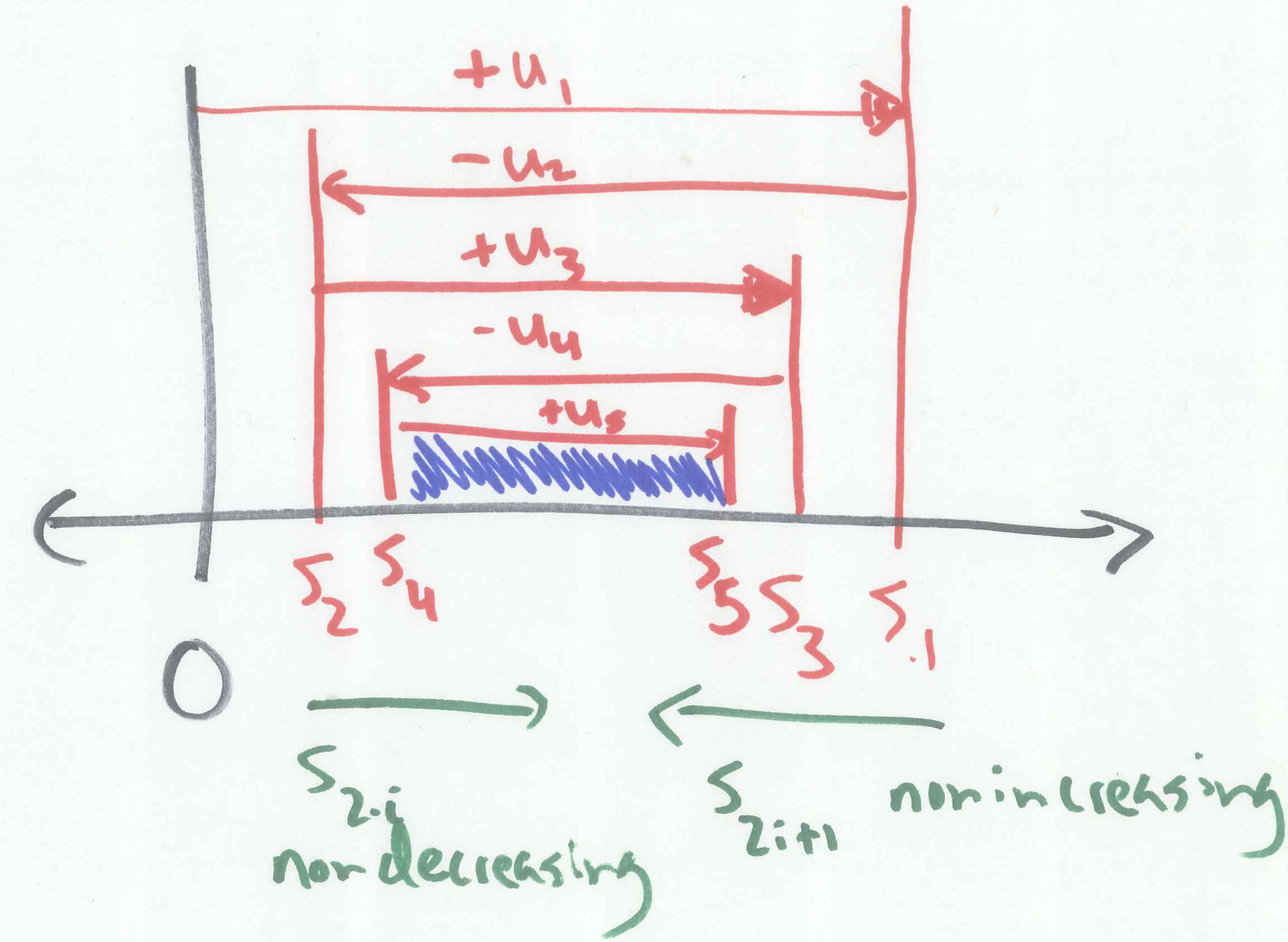
Let

$$\sum_{n=1}^{\infty} (-1)^{n+1} u_n = u_1 - u_2 + u_3 - u_4 \cdots$$

be an alternating series.
This series converges if all three of the following are true:

1. $u_n > 0$.
2. $\{u_n\}$ is nonincreasing (eventually): $u_n \geq u_{n+1}$ for all $n \geq N$.
3. $u_n \to 0$. (n^th term test doesn't show divergence)

Idea:



$+u_1$
$-u_2$
$+u_3$
$-u_4$
$+u_5$
s_2 s_4 s_5 s_3 s_1
0
$s_{2 \cdot i}$ nondecreasing
s_{2i+1} nonincreasing

Converges		Diverges
Absolutely Converges	Conditionally Converges	Diverges
$\sum \lvert a_n \rvert$ converges	$\sum a_n$ converges but $\sum \lvert a_n \rvert$ diverges	$\sum a_n$ diverges

a_n must be both positive and negative

Ex: $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$

$u_n = \frac{1}{n} > 0$ 1. ✓

$\frac{1}{n} \geq \frac{1}{n+1}$ 2. ✓

$\frac{1}{n} \to 0$ 3. ✓

CONDITIONALLY CONVERGES

$\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2}$

$u_n = \frac{1}{n^2} > 0$ 1. ✓

$\frac{1}{n^2} \geq \frac{1}{(n+1)^2}$ 2. ✓

$\frac{1}{n^2} \to 0$ 3. ✓

$\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n}}{n!}$

$u_n = \frac{1}{n!}$

ABSOLUTELY CONVERGES

$\displaystyle\sum_{n=1}^{\infty} \underline{\underline{(-1)^n}} \underline{\left(1 - \frac{1}{n}\right)^n}$

$u_n = \left(1 - \frac{1}{n}\right)^n \geq 0$ 1. ✓

2. ✗ ?

$u_n \to e^{-1}$ 3. ✗

$\displaystyle\sum_{n=1}^{\infty} (-1)^n \sin n$

$u_n = \sin n$ 1. ✗

Thm 15 (Alt. Series Estimation)

If $\sum_{n=1}^{\infty} (-1)^{n+1} u_n = S$, and

$$s_N = \sum_{n=1}^{N} (-1)^{n+1} u_n, \quad \text{then}$$

$$|S - s_N| \leq u_{N+1} \; (< \varepsilon)$$

WANT

Ex: Consider $\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}$.

How many term must I sum to approximate the value within $\underbrace{10^{-2}}_{\varepsilon}$?

u_{N+1}

WANT: $\underline{\frac{1}{(N+1)^2}} < 10^{-2}$

$(N+1)^2 > 10^2$

$N+1 > 10$

$N > 9$

($N=10$ suffices)

$S_n = \underbrace{\frac{-1}{1} + \frac{1}{4} - \frac{1}{9} + \frac{1}{16} - \frac{1}{25} + \frac{1}{36} - \frac{1}{49} \cdots + \frac{1}{100}}_{S_{10}}$

Thm 16 (Absolute Convergence Test)

If $\sum_{n=1}^{\infty} |a_n|$ converges,

then $\sum_{n=1}^{\infty} a_n$ converges.

Let b_n be positive terms of a_n

c_n be negative terms

$b_n = \max\{a_n, 0\}$

$c_n = \min\{a_n, 0\}$

$\sum |a_n|$ converges $\rightarrow$ $\sum b_n$ converges, $\sum |c_n|$ converges

$$\sum b_n + \sum c_n = \sum (b_n + c_n) = \sum a_n$$

THE REVERSE IS NOT TRUE!

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \quad \text{converges, but}$$

$$\sum_{n=1}^{\infty} \frac{1}{n} \quad \underline{\text{diverges}}$$

BOTH MUST CONVERGE!

$$\sum a_n - \sum b_n = \sum (a_n - b_n)$$

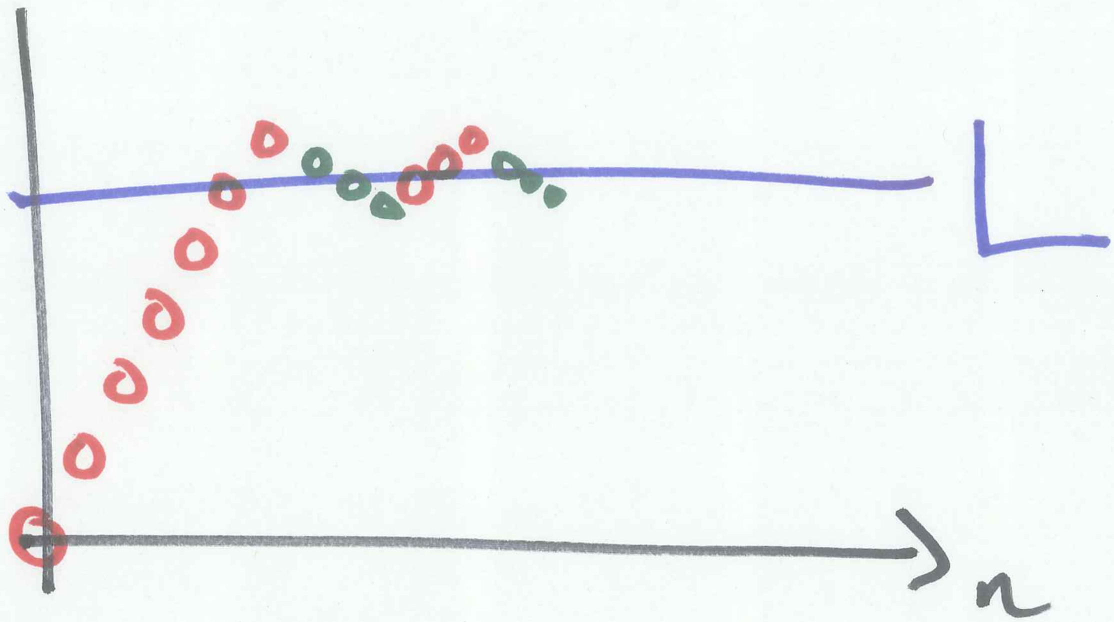
Thm 17 (Rearrangement Thm):

If $\sum a_n$ converges absolutely and $\{b_n\}_{n=1}^{\infty}$ is a rearrangement of the sequence $\{a_n\}$, then

$$\sum a_n = \sum b_n .$$

If $\sum a_n$ converges conditionally and L is a real number, then there exists a rearrangement $\{b_n\}$ of the sequence $\{a_n\}$ such that

$$\sum b_n = L .$$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