10.7: Power Series

Polynomial: $p(x) = c_0 + c_1 x + c_2 x^2 + \cdots + c_n x^n$

Power Series: $f(x) = \sum_{n=0}^{\infty} c_n x^n$
(Centered ~~at~~ 0)

Power Series: $f(x) = \sum_{n=0}^{\infty} c_n (x-a)^n$
(Centered at a)

Ex: $\dfrac{1}{1-x} = \sum_{n=0}^{\infty} x^n \quad (|x| < 1)$

What it converges to

Where it converges

Ex: For what values does the power series

$$\sum_{n=1}^{\infty} \frac{x^n}{n}$$

converge?

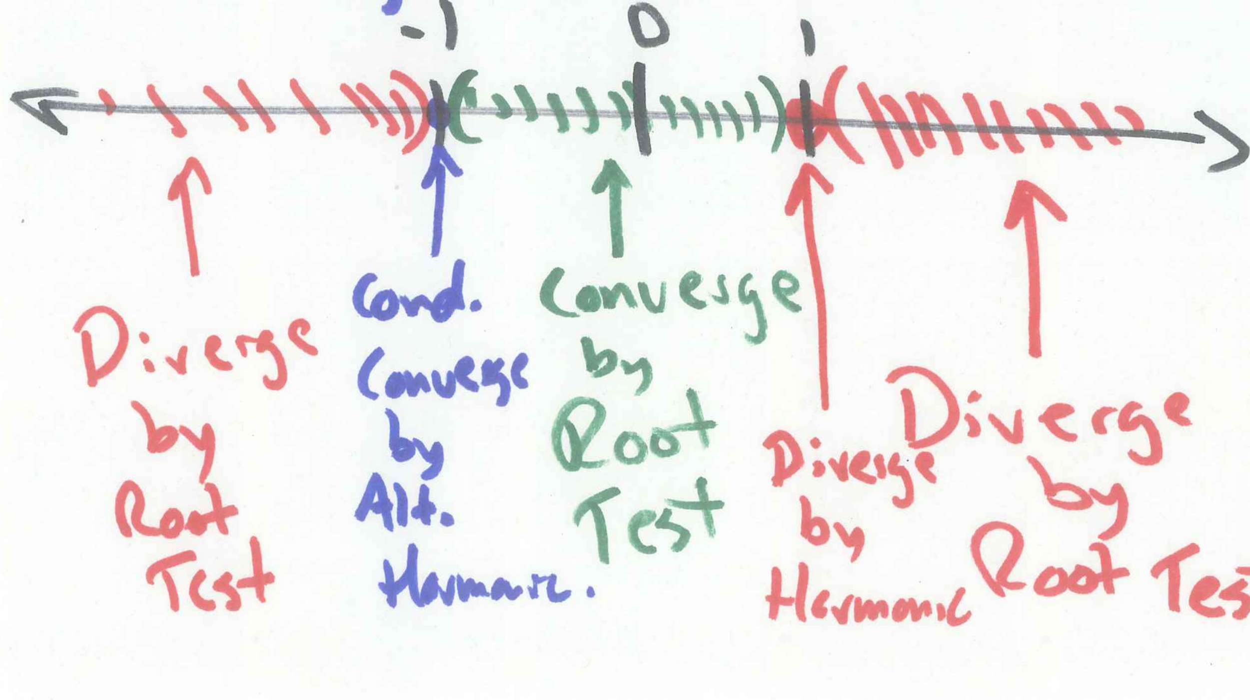
$x=1 \rightarrow \sum \frac{1}{n}$ $\qquad$ $x=-1 \rightarrow \sum \frac{(-1)^n}{n}$

important!

Root Test: $\lim_{n\to\infty} \sqrt[n]{\frac{|x^n|}{n}} = \lim_{n\to\infty} \frac{|x|}{\sqrt[n]{n}} = |x|$

If $|x| < 1 \rightarrow$ CONVERGE!

If $|x| > 1 \rightarrow$ DIVERGE!



Ex: For what values does the power series

$$\sum_{n=0}^{\infty} (x+1)^n \qquad a = -1 \text{ (center)}$$

converge?

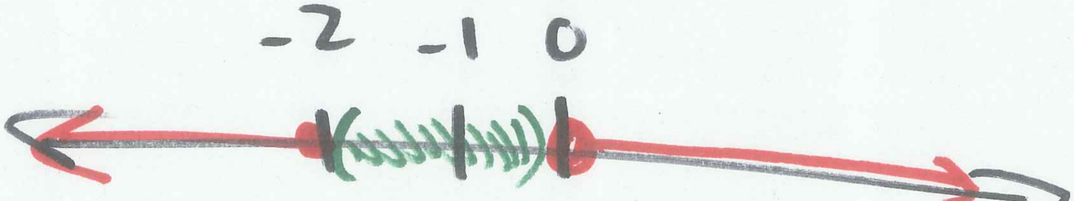
Converge iff $|x+1| < 1$

$$\Updownarrow$$

$$\begin{array}{ccccc} -1 & < & x+1 & < & 1 \\ -1 & & -1 & & -1 \end{array}$$

$$\Updownarrow$$

$$-2 < x < 0$$



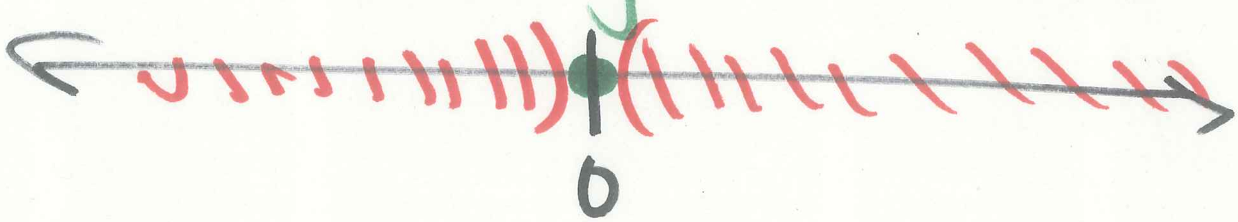
Ex: $\displaystyle\sum_{n=0}^{\infty} n!\,x^n$

Ratio Test: $\displaystyle\lim_{n\to\infty} \frac{(n+1)\cancel{!}\,|x|^{\cancel{n+1}}}{\cancel{n!}\,\cancel{|x|^{n}}}$

$$\Vert$$

$$\lim_{n\to\infty} (n+1)\cdot|x| = \begin{cases} 0 & \text{if } x=0 \\ \infty & \text{if } x\neq 0 \end{cases}$$

The series converges at $x=0$,
Diverges otherwise.



Thm 18: (The Convergence Thm for Power Series) If the power series

$$\sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + \cdots$$

converges at $x=b$ for some $b \neq 0$, then the series converges for all x with $|x| < |b|$.

- If it diverges at $x=d$, then it diverges for all x with

$$|x| > |d|.$$

(picture)

Cor: The convergence of $\sum_{n=0}^{\infty} C_n (x-a)^n$ is described by one of these cases:

1: There is a positive number R such that the series

- converges for x in $(a-R, a+R)$
- diverges for x in $(-\infty, a-R)$ or in $(a+R, \infty)$
- may or may not converge at $a+R$ and $a-R$

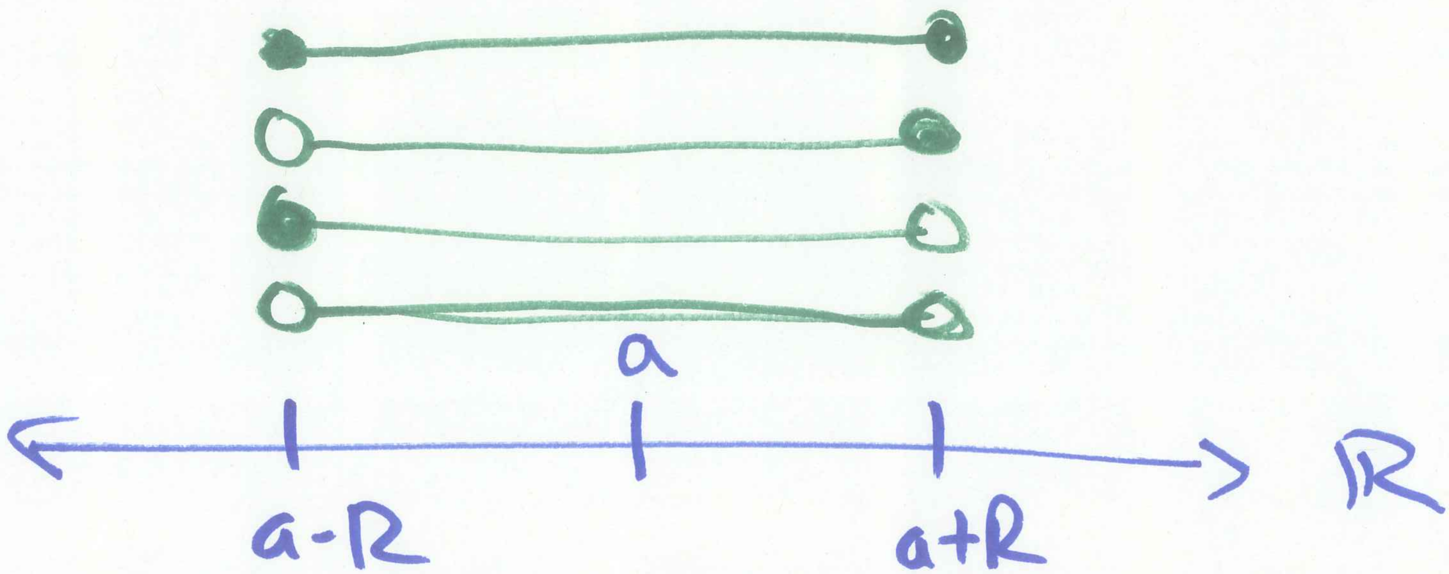
2: Converges absolutely for all x $(R=\infty)$

3:
- Converges at $x=a$
- Diverges at $x \neq a$.

$(R=0)$

Def: R is the <u>radius of convergence</u>

The interval of radius R centered at a is the <u>interval of convergence</u>.



Ex: $\displaystyle\sum_{n=1}^{\infty} \frac{n x^n}{5^n}$ $\qquad$ $x=5$: $\displaystyle\sum_{n=1}^{\infty} n$ $\qquad$ $x=-5$: $\displaystyle\sum_{n=1}^{\infty} (-1)^n n$

Diverge by n^{th} term test

Root Test: $\displaystyle\lim_{n\to\infty} \sqrt[n]{\frac{n|x|^n}{5^n}} = \lim_{n\to\infty} \sqrt[n]{n}\, \frac{|x|}{5}$

$$= \frac{|x|}{5}$$

Need: $5 \cdot \dfrac{|x|}{5} < 1 \cdot 5$

$$\Updownarrow$$

$|x| < 5 \leftarrow R$

Converges Absolutely on $(-5, 5)$

Diverges in $(-\infty, -5) \cup (5, \infty)$

Diverges for $x=-5$ and $x=5$

Interval of Convergence: $(-5, 5)$

Ex: $\displaystyle\sum_{n=0}^{\infty} \frac{x^n}{n!}$

Ratio Test:

$$\lim_{n\to\infty} \frac{|x|^{\cancel{n}+1}}{(n+1)\cancel{!}} \cdot \frac{\cancel{n!}}{\cancel{|x|^n}}$$

$$= \lim_{n\to\infty} \frac{|x|}{n+1} = 0 < 1 \text{ for } \underline{\text{all}}\ x!$$

$R = \infty$

Interval of Convergence: $(-\infty, \infty)$

Ex: $\sum_{n=1}^{\infty} \frac{x^n}{n^2}$

Root Test: $\lim_{n\to\infty} \sqrt[n]{\frac{|x|^n}{n^2}}$

$= \lim_{n\to\infty} \frac{|x|}{\sqrt[n]{n^2}} = |x|$

If $|x|<1$, converge absolutely,

so $\underline{R=1}$

$x=1$: $\sum_{n=1}^{\infty} \frac{1}{n^2}$ Converges by p-test

$x=-1$: $\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}$ absolutely converges!

Interval of Convergence: $[-1,1]$

How to Test a Power Series for Convergence

1. Use the Ratio Test (or Root Test) to find interval where it converges absolutely:

$$|x-a| < R$$

$$a-R < x < a+R$$

2. If $R < \infty$, test for convergence or divergence at end points of interval: $x = a-R$ and $x = a+R$

3. If $R < \infty$, then the series diverges if $x < a-R$ or $x > a+R$.