Operations on a Power Series

Let $f(x) = \sum_{n=0}^{\infty} c_n (x-a)^n$

be a power series that absolutely converges when $|x-a| < R$.

- $f(a) = c_0$

- $\frac{d}{dx} f(x) = \frac{d}{dx} \sum_{n=0}^{\infty} c_n (x-a)^n$

$$= \sum_{n=0}^{\infty} \frac{d}{dx} c_n (x-a)^n$$

Differentiating a Power Series

$$\frac{d}{dx}\sum_{n=0}^{\infty} c_n (x-a)^n \leftarrow f(x)$$

$$= \sum_{n=0}^{\infty}\left[\frac{d}{dx} c_n (x-a)^n\right]$$

$$= \sum_{n=\cancel{0}\,1}^{\infty} c_n \cdot n (x-a)^{n-1} \leftarrow \frac{d}{dx}f(x) = f'(x)$$

- $f(a) = c_0$
- $f'(a) = c_1$
- $f''(a) = 2c_2$

$$\underbrace{f^{(n)}(a) = n! \cdot c_n}_{!!!!!!!}$$

Integrating a Power Series

$$\int \left[\sum_{n=0}^{\infty} c_n (x-a)^n \right] dx$$

$$= \sum_{n=0}^{\infty} \left[\int c_n (x-a)^n dx \right]$$

$$= \sum_{n=0}^{\infty} \left[\frac{c_n}{n+1} (x-a)^{n+1} \right] + C$$

unknown constant

Ex: $g(x) = \sum_{n=0}^{\infty} x^n = \frac{1}{1-x}$

$$\frac{d}{dx} g(x) = \frac{d}{dx} \sum_{n=0}^{\infty} x^n$$

$$= \sum_{n=0}^{\infty} \frac{d}{dx} x^n$$

$$= \sum_{n=1}^{\infty} n \cdot x^{n-1} = \sum_{m=0}^{\infty} (m+1) x^m$$

$m = n-1$

$$\frac{d}{dx}\left[\frac{1}{1-x}\right] = \frac{d}{dx}\left[(1-x)^{-1}\right]$$

$$= \cancel{(-1)}(1-x)^{-2} \cdot (-1)$$

$$= \frac{1}{(1-x)^2} = \sum_{m=0}^{\infty} (m+1) x^m$$

Ex: $h(x) = \sum_{n=0}^{\infty} n \cdot x^n$

$$= \sum_{n=0}^{\infty} \left((n+1) - 1 \right) x^n$$

$$= \sum_{n=0}^{\infty} (n+1) x^n - \sum_{n=0}^{\infty} x^n$$

Same Radius of Convergence

$$= \frac{1}{(1-x)^2} - \frac{1}{1-x}$$

Ex: $f(x) = \sum_{n=0}^{\infty} \frac{x^n}{n!}$

$$\frac{d}{dx} f(x) = \sum_{n=0}^{\infty} \frac{d}{dx} \frac{x^n}{n!}$$

$$= \sum_{n=1}^{\infty} \frac{n \cdot x^{n-1}}{n!}$$

$$= \sum_{n=1}^{\infty} \frac{x^{n-1}}{(n-1)!}$$

$$= \sum_{m=0}^{\infty} \frac{x^m}{m!} = f(x) = \underline{\underline{e^x}}$$

$$\boxed{\sum_{n=0}^{\infty} \frac{1}{n!} = e}$$

10.8: Taylor & Maclaurin Series

Suppose

$$f(x) = \sum_{n=0}^{\infty} c_n (x-a)^n$$

$$= c_0 + c_1(x-a)^1 + c_2(x-a)^2 + c_3(x-a)^3 + \cdots$$

Then

$$f'(x) = 0 + 1 \cdot c_1 + 2c_2(x-a)^1 + 3c_3(x-a)^2$$

power series

$$f''(x) = 0 + 2 \cdot 1 \cdot c_2 + 3 \cdot 2 \cdot c_3 (x-a)^1 + \cdots$$

$$f^{(3)}(x) = 0 + 3 \cdot 2 \cdot 1 \cdot c_3 + \cdots$$

$$f^{(n)}(a) = n! \cdot c_n \rightarrow c_n = \frac{f^{(n)}(a)}{n!}$$

Idea:

If we define

$$h(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$$

then this satisfies

$$h^{(n)}(a) = f^{(n)}(a) \quad *$$

for all $n \geq 0$.

(* If this makes sense.)

Also: $h(x) = f(x)$
(wherever $h(x)$ makes sense)

Ex: $f(x) = e^x$

Center at $a = 0$

$$f^{(n)}(x) = e^x$$

$$f^{(n)}(0) = e^0 = 1.$$

$$e^x = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

Def: Let $f(x)$ be a function with derivatives $f^{(n)}(a)$ all defined.

The Taylor Series generated by f at $x=a$ is

$$\sum_{k=0}^{\infty} \frac{f^{(k)}(a)}{k!}(x-a)^k = f(x)$$

wherever the power series converges.

$$= f(a) + f'(a)(x-a) + \frac{f''(a)}{2}\cdot(x-a)^2 + \frac{f'''(a)}{6}\cdot(x-a)^3 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n + \cdots$$

(If $a=0$, we call this a Maclaurin Series)

Ex: $f(x) = \sin x \xmapsto{x=0} 0$

$f'(x) = \cos x \longmapsto 1$

$f''(x) = -\sin x \longmapsto 0$

$f^{(3)}(x) = -\cos x \longmapsto -1$

$f^{(4)}(x) = \sin x \longmapsto 0$

$$\sin x = \sum_{i=0}^{\infty} \frac{(-1)^i x^{2i+1}}{(2i+1)!}$$

$(n = 2i+1)$

$$\cos x = \sum_{i=0}^{\infty} \frac{(-1)^i x^{2i}}{(2i)!}$$