Ex: $f(x) = 3 + 2x + 4x^2 + x^3 + 0x^4 + 0x^5 + \cdots$

$(a=0)$

$f^{(n)}(0) = n! \cdot c_n,$

$f(0) = 3$

$f'(0) = 2$

$f''(0) = 8$

$f'''(0) = 6$

$(a=1)$ $f(x) = 3 + 2x + 4x^2 + x^3$

$f(1) = 3+2+4+1 = 10$

$f'(1) = 13$ $\qquad f'(x) = 2 + 8x + 3x^2$

$f''(1) = 14$ $\qquad f''(x) = 8 + 6x$

$f'''(1) = 6$ $\qquad f'''(x) = 6$

$$f(x) = \underset{\uparrow \atop \frac{f(a)}{0!}}{\underline{10}} + \underset{\uparrow \atop \frac{f'(a)}{1!}}{\underline{13}}(x-1) + \underset{\uparrow \atop \frac{f''(a)}{2!}}{\underline{7}}(x-1)^2 + \underset{\uparrow \atop \frac{f'''(a)}{3!}}{\underline{1}}(x-1)^3$$

Ex: $f(x) = \ln x$ Center: $a = 1$

$f^{(1)}(x) = \frac{1}{x} = x^{-1}$ $\quad f(1) = 0$

$f'(1) = \frac{1}{1} = 1$

$f^{(2)}(x) = \frac{-1}{x^2} = -1 \cdot x^{-2}$ $\quad f^{(2)}(1) = \frac{-1}{1} = -1$

$f^{(3)}(x) = \frac{2}{x^3} = 2 \cdot x^{-3}$ $\quad f^{(3)}(1) = \frac{2}{1} = 2$

$f^{(4)}(x) = \frac{-3 \cdot 2}{x^4} = -3 \cdot 2 \cdot x^{-4}$ $\quad f^{(4)}(1) = -6 = -3!$

$f^{(n)}(x) = (-1)^{n+1} (n-1)!\, x^{-n}$ $\quad f^{(n)}(1) = \underline{\underline{(-1)^{n+1}(n-1)!}}$

$$\ln x = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}(n-1)!}{n!}(x-1)^n$$

$$\ln x = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}(x-1)^n$$

Does not converge everywhere!

Root Test: $\lim_{n\to\infty} \sqrt[n]{\frac{1}{n}|x-1|^n} = \underline{|x-1|} < 1$

$\Rightarrow -1 < x-1 < 1 \Rightarrow \underline{0 < x < 2}$

$$\ln x = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}(x-1)^n \qquad 0 < x \le 2$$

$$\frac{d}{dx}\ln x = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{\cancel{n}} \cdot \cancel{n}(x-1)^{n-1}$$

$$= \sum_{n=1}^{\infty} (-1)^{n+1} \cdot (x-1)^{n-1}$$

$$(-1)^{m+2} = (-1)^m \cdot (-1)^2$$

$$= \sum_{m=0}^{\infty} \underbrace{(-1)^m} \cdot \underbrace{(x-1)^m} \qquad (m = n-1)$$

$$= \sum_{m=0}^{\infty} (1-x)^m \qquad \left(\begin{array}{l}\text{Converges} \\ \text{when } |1-x| < 1\end{array}\right)$$

$$= \frac{1}{1-(1-x)} = \underline{\underline{\frac{1}{x}}}$$

$$\ln(x+1) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} x^n$$

Ex: $f(x) = e^{-x}$

$$e^y = \sum_{n=0}^{\infty} \frac{y^n}{n!}$$

$$e^{-x} = \sum_{n=0}^{\infty} \frac{(-x)^n}{n!} = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} x^n$$

$g(x) = e^{x^2}$

$$e^{x^2} = \sum_{n=0}^{\infty} \frac{(x^2)^n}{n!} = \sum_{n=0}^{\infty} \frac{x^{2n}}{n!}$$

Ex: $\left(1+\frac{x}{n}\right)^n$

n choose 2 $\binom{n}{2}$ n choose 3 $\binom{n}{3}$

$$= 1 + (n)\frac{x}{n} + \frac{n(n-1)}{2}\frac{x^2}{n^2} + \frac{n(n-1)(n-2)}{3\cdot 2\cdot 1}\frac{x^3}{n^3}$$

$$= 1 + x + \frac{n(n-1)}{n^2}\frac{x^2}{2!} + \frac{n(n-1)(n-2)}{n^3}\cdot\frac{x^3}{3!} + \cdots$$

$$\Big\downarrow {\scriptstyle n\to\infty}$$

$$\sum_{n=0}^{\infty}\frac{x^n}{n!} = e^x$$

$$\sin x = \sum_{j=0}^{\infty} \frac{(-1)^j x^{2j+1}}{(2j+1)!}$$

$$\cos x = \sum_{j=0}^{\infty} \frac{(-1)^j x^{2j}}{(2j)!}$$

$$e^{x} = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

$$e^{-x} = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} x^n$$

$$e^{i\theta} = \sum_{n=0}^{\infty} \frac{i^n \cdot \theta^n}{n!}$$

$$= \sum \frac{(-1)^j \theta^{2j}}{(2j)!} + \sum \frac{(-1)^j i\, \theta^{2j+1}}{(2j+1)!}$$

$$= \cos\theta + i \sin\theta$$

$$e^{i\pi} = \cos\pi + i\sin\pi$$
$$= -1$$

$$e^{i\pi} = -1$$

$$\boxed{e^{i\pi} + 1 = 0}$$