$$f(x) = \sqrt{1+x^3+\sin(x)}$$

$$= \left(1+x^3+\sin(x)\right)^{1/2}$$

$$f'(x) = \frac{1}{2}\left(1+x^3+\sin x\right)^{-1/2}\left(3x^2+\cos x\right)$$

$$\vdots$$

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n \qquad (a=0)$$

$$f(0) = \sqrt{1+0^3+\sin 0} = 1$$

$$f'(0) = \frac{1}{2}\underbrace{\left(1+0^3+\sin 0\right)^{-1/2}}_{1}\underbrace{\left(3\cdot 0^2+\cos 0\right)}_{1}$$

$$= \frac{1}{2}$$

$$f(x) = \underbrace{1+\frac{1}{2}x}_{\text{Linear approximation}} + \cdots \rightarrow ?$$

10.9: Convergence of Taylor Series

Def: The n^{th} Taylor polynomial of $f(x)$ centered at $x=a$ is

$$P_n(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n$$

$$= \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}(x-a)^k$$

Ex: Determine the 3rd Taylor Polynomial for $f(x) = \sin^2 x$ centered at $x=0$

$$f'(x) = 2\sin x \cdot \cos x$$

$$f''(x) = 2\left[\cos^2 x - \sin^2 x\right]$$

$$f^{(3)}(x) = 2\left[2\cos x \cdot (-\sin x) - 2\sin x \cos x\right]$$

$f(0) = 0$, $f'(0) = 0$, $f''(0) = 2$, $f^{(3)}(0) = 0$

3rd Taylor Polynomial: $\dfrac{2}{2!}x^2 = \underline{\underline{x^2}}$

Taylor's Formula

If f has derivatives of all orders on an open interval I containing a, then for each $n \geq 1$ and all x in I,

$$f(x) = f(a) + f'(a)(x-a) + f''(a)\frac{(x-a)^2}{2!}$$

$$+ \cdots + f^{(n)}(a)\frac{(x-a)^n}{n!} + R_n(x)$$

where

$$R_n(x) = \frac{f^{(n+1)}(c)}{(n+1)!}(x-a)^{n+1}$$

for some c between a and x.

Thm 24 (The Remainder Est. Thm)

If there is a constant M s/t

$$\left|f^{(n+1)}(t)\right| \leq M \text{ for all } t$$

between x and a, then

$$\left|R_n(x)\right| \leq M \cdot \frac{|x-a|^{n+1}}{(n+1)!}$$

Replaces

$f^{(n+1)}(c)$

(for some c)

Ex: Use Taylor's Formula to show that

$$\sin x = \sum_{j=0}^{\infty} \frac{(-1)^j x^{2j+1}}{(2j+1)!}$$

$$P_{2n+1}(x) = \sum_{j=0}^{n} \frac{(-1)^j x^{2j+1}}{(2j+1)!}$$

Taylor's Formula:

want to go to 0 as $n \to \infty$

$$\sin x = P_{2n+1}(x) + R_{2n+1}(x)$$

Need to control $f^{(2n+2)}(t) = (-1)^{?} \sin t$

$$|f^{(2n+2)}(t)| \leq 1 = M$$

Est. Thm: $R_{2n+1}(x) \leq M \cdot \dfrac{|x-0|^{2n+2}}{(2n+2)!}$

$$= \frac{|x|^{2n+2}}{(2n+2)!} \xrightarrow[\text{as } n \to \infty]{} 0$$

Ex: For what values of x can we replace e^x by $1+x+x^2/2$ with an error of magnitude no greater than 2×10^{-2}?

Taylor's Formula: $e^x = \underbrace{1+x+\frac{x^2}{2}}_{P_2(x)} + R_2(x)$

Q: For what x is $|R_2(x)| \leq \frac{2}{100}$?

Est Thm: $|R_2(x)| \leq \underbrace{M\cdot\frac{|x|^3}{3!} \leq \frac{2}{100}}_{\text{for what } x \text{ is this true?}}$

$$f^{(3)}(x) = e^x \leq e^{|x|}$$

$$|R_2(x)| \leq \underbrace{\frac{e^{|x|}\cdot|x|^3}{6} \leq \frac{2}{100}}_{\text{To Finish: Set equal \& "solve"}}$$

Using Power Series Operations to construct Taylor Series

- Substitution
- Differentiation
- Integration
- Addition / Subtraction
- Multiplication by x^p (or $(x-a)^p$)
- Trig Identities

$$f(x) = \frac{1}{1+x} = \frac{1}{1-(-x)} = g(-x)$$

$$g(x) = \frac{1}{1-x} = \sum_{n=0}^{\infty} x^n \qquad (|x|<1)$$

$$\frac{1}{1+x} = \sum_{n=0}^{\infty} (-x)^n = \sum_{n=0}^{\infty} (-1)^n x^n$$

Substitution

$$f(x) = x \cdot e^{x}$$

$$e^{x} = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2} + \frac{x^3}{3!} + \cdots$$

$$xe^{x} = \sum_{n=0}^{\infty} \frac{x^{n+1}}{n!} = x + x^2 + \frac{x^3}{2} + \frac{x^4}{3!} + \cdots$$

$f(x) = x^2 \cdot \sin x$

$$f(x) = \frac{1}{(1-x)^2}$$

$$f(x) = \arctan(x)$$

$$f'(x) = \frac{1}{x^2+1} = \frac{1}{1-(-x^2)} = g(-x^2)$$

$$g(x) = \frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$$

$$f'(x) = g(-x^2) = \sum_{n=0}^{\infty} (-x^2)^n = \sum_{n=0}^{\infty} (-1)^n x^{2n}$$

$$\arctan x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1} \quad (+C)$$

$$\underline{C=0}$$

Frequently Used Taylor Series

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n \qquad (|x| < 1)$$

$$\frac{1}{1+x} = \sum_{n=0}^{\infty} (-1)^n x^n \qquad (|x| < 1)$$

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

$$\sin x = \sum_{j=0}^{\infty} \frac{(-1)^j x^{2j+1}}{(2j+1)!}$$

$$\cos x = \sum_{j=0}^{\infty} \frac{(-1)^j x^{2j}}{(2j)!}$$

$$\ln(1+x) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1} x^n}{n} \qquad (-1 < x \leq 1)$$

$$\arctan(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2n+1} \qquad (|x| \leq 1)$$