$$f(x) = \frac{1}{(1-x)^2} = \frac{1}{(1-x)(1-x)}$$ (Find a power series representation)

$$g(x) = \frac{1}{(1-x)^1} = (1-x) \cdot f(x)$$

$$f(x) = (1-x)^{-2} \qquad \int f(x)\,dx = (+1)(1-x)^{-1}$$

$$g(x) = (1-x)^{-1}$$

$$\frac{d}{dx} g(x) = (-1)(1-x)^{-2}(-1) = (1-x)^{-2} = f(x)$$

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$$

$\frac{d}{dx}$ $\frac{d}{dx}$ term-wise

$$\frac{1}{(1-x)^2} = \sum_{n=1}^{\infty} n x^{n-1} = \sum_{m=0}^{\infty} (m+1) x^m$$

Section 10.10: Binomial Series and Applications of Taylor Series

"m choose k"

Binomial Coefficient $\binom{m}{k}$

(m real, k integer, $k \geq 0$)

$$\binom{m}{k} = \frac{m(m-1)\cdots \overbrace{(m-k+1)}^{= m-(k-1)}}{k(k-1)\cdots 3\cdot 2\cdot 1} \begin{array}{l} \leftarrow k \text{ terms} \\ \leftarrow k \text{ terms} \end{array}$$

Ex: $\binom{m}{1} = m$

$$\binom{m}{2} = \frac{m(m-1)}{2}$$

$$\binom{m}{0} = 1 = \frac{1}{0!}$$

Ex: $(1+x)^4$

$$= (1+x)\cdot(1+x)\cdot(1+x)\cdot(1+x)$$

$$= \underset{\binom{4}{0}}{\underset{\uparrow}{1}} + \underset{\binom{4}{1}}{\underset{\uparrow}{4}}x + \underset{\binom{4}{2}}{\underset{\uparrow}{6}}x^2 + \underset{\binom{4}{3}}{\underset{\uparrow}{4}}x^3 + \underset{\binom{4}{4}}{\underset{\uparrow}{1}}x^4$$

Pascal's Triangle ($\binom{m}{k}$ for integer m)

$$
\begin{array}{ccccccccccccccc}
 & & & & & & & 1 & & & & & & & \\
 & & & & & & 1 & & 1 & & & & & & \\
 & & & & & 1 & & 2 & & 1 & & & & & \\
 & & & & 1 & & 3 & & 3 & & 1 & & & & \\
 & & & 1 & & 4 & & 6 & & 4 & & 1 & & & \\
 & & 1 & & 5 & & 10 & & 10 & & 5 & & 1 & & \\
 & 1 & & 6 & & 15 & & 20 & & 15 & & 6 & & 1 & \\
1 & & 7 & & 21 & & 35 & & 35 & & 21 & & 7 & & 1
\end{array}
$$

m

k

Binomial: $(1+x)^m$

If m is an integer, then this is a polynomial.

$$f(x) = (1+x)^m \qquad f(0) = 1$$

$$f'(x) = m\cdot(1+x)^{m-1} \qquad f'(0) = m$$

$$f''(x) = m\cdot(m-1)(1+x)^{m-2} \qquad f''(0) = m(m-1)$$

$$f^{(3)}(x) = m\cdot(m-1)\cdot(m-2)(1+x)^{m-3} \qquad f^{(3)}(0) = m(m-1)(m-2)$$

$$\vdots \qquad \vdots$$

$$f^{(k)}(x) = m\cdot(m-1)\cdots(m-k+1)(1+x)^{m-k} \qquad f^{(k)}(0) = k!\cdot\binom{m}{k}$$

$$\| $$

$$m\cdot(m-1)\cdots(m-k+1)(1+x)^{m-k}$$

The Binomial Series:

For $-1 < x < 1$,

$$(1+x)^m = \sum_{k=0}^{\infty} \binom{m}{k} x^k$$

$$= 1 + \sum_{k=1}^{\infty} \binom{m}{k} x^k$$

where

$$\binom{m}{k} = \frac{m\cdot(m-1)\cdot(m-2)\cdots(m-k+1)}{k!}$$

and $\binom{m}{0} = \frac{1}{0!} = 1$

Ex: $(1+x)^{3/2} = \left(\sqrt{1+x}\right)^3$

$m = 3/2$

$$(1+x)^{3/2} = \sum_{k=0}^{\infty} \binom{3/2}{k} x^k$$

$$= \binom{3/2}{0} + \binom{3/2}{1} x^1 + \binom{3/2}{2} x^2 + \binom{3/2}{3} x^3 + \cdots$$

$$= 1 + \frac{3}{2}x + \left(\frac{\frac{3}{2}\cdot\frac{1}{2}}{2}\right) x^2 + \left(\frac{\frac{3}{2}\cdot\frac{1}{2}\cdot\frac{-1}{2}}{3\cdot 2\cdot 1}\right) x^3 + \cdots$$

$$= 1 + \frac{3}{2}x + \frac{3}{8}x^2 - \frac{1}{16}x^3 + \cdots$$

Ex: $(1+x)^{-1} = \frac{1}{1+x} = \frac{1}{1-(-x)}$

$$(1+x)^{-1} = \sum_{k=0}^{\infty} \binom{-1}{k} \cdot x^k$$

$$= \binom{-1}{0} + \binom{-1}{1}x + \binom{-1}{2}x^2 + \binom{-1}{3}x^3 + \cdots$$

$$= \underbrace{1}_{1} + (\underbrace{-}_{-1}x) + \underbrace{\frac{(-1)\cdot(-2)}{2\cdot 1}}_{1} x^2 + \underbrace{\frac{(-1)(-2)(-3)}{3\cdot 2\cdot 1}}_{-1} x^3$$

Note: $\binom{-1}{k} = (-1)^k$

$$(1+x)^{-1} = \sum_{k=0}^{\infty} (-1)^k x^k = \frac{1}{1+x}$$

Application: Evaluating Non-Standard Integrals

Express $\int \cos(x^2)\,dx$ as a power series.

$$\cos y = \sum_{j=0}^{\infty} \frac{(-1)^j y^{2j}}{(2j)!}$$

$$\cos(x^2) = \sum_{j=0}^{\infty} \frac{(-1)^j (x^2)^{2j}}{(2j)!} = \sum_{j=0}^{\infty} \frac{(-1)^j x^{4j}}{(2j)!}$$

$$\int \cos(x^2)\,dx = C + \sum_{j=0}^{\infty} \frac{(-1)^j}{(2j)!} \cdot \frac{x^{4j+1}}{4j+1}$$

Application: Evaluating Indeterminate Forms

Compute the following limits:

$$\lim_{x \to 1} \frac{\ln x}{x-1}$$

$$\lim_{x \to 0} \frac{\sin x - \tan x}{x^3}$$

$$\lim_{x \to 0} \left(\frac{1}{\sin x} - \frac{1}{x} \right)$$

* $$\lim_{x \to 0} \frac{e^x - (1+x)}{x^2}$$

$$\lim_{x \to 0} \frac{e^x - (1+x)}{x^2}$$

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

$$= 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \cdots$$

$$= \lim_{x \to 0} \frac{\frac{x^2}{2} + \frac{x^3}{6} + \cdots}{x^2}$$

$$= \lim_{x \to 0} \frac{1}{2} + \frac{x}{6} + \cdots$$

$$= \underline{\frac{1}{2}}$$

Ex: Determine

$$\lim_{x \to 0} \frac{\ln(1+x^2)}{1-\cos x}$$

$$\ln(1+x) = \sum_{n=1}^{\infty} \frac{x^n \cdot (-1)^{n-1}}{n}$$

$$\ln(1+x^2) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} (x^2)^n$$

$$= x^2 - \frac{x^4}{2} + \frac{x^6}{3} - \cdots$$

$$\cos x = \sum_{j=0}^{\infty} \frac{(-1)^j}{(2j)!} x^{2j}$$

$$= 1 - \frac{x^2}{2} + \frac{x^4}{4!} - \frac{x^6}{6!}$$

$$1 - \cos x = 0 + \frac{x^2}{2} - \frac{x^4}{4!} + \frac{x^6}{6!} \cdots$$

$$\lim_{x \to 0} \frac{x^2 - \frac{x^4}{2} + \frac{x^6}{3} + \cdots}{\frac{x^2}{2} - \frac{x^4}{4!} + \frac{x^6}{6!} + \cdots}$$

$$= \lim_{x \to 0} \frac{1 - \frac{x^2}{2} + \frac{x^4}{3} - \cdots}{\frac{1}{2} - \frac{x^2}{4!} + \frac{x^6}{6!} + \cdots} = \underline{\underline{\underline{2}}}$$