Series Review

Series Tests

n^{th}-term
p-Series, Geometric, Telescoping
Comparison Tests
Ratio/Root Tests
Integral Test
Alternating Series Test

Power Series

$$\sum_{n=0}^{\infty} c_n (x-a)^n$$

Q: What is the center?

Q: Radius of Convergence?

Q: Interval of Convergence?

Root Test / Ratio Test

on $\sum |c_n| \cdot |x-a|^n$

On endpoints,

Use Other Tests!

Taylor Series

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$$

Q: What are the first few terms?

(What is the n^{th} Taylor Polynomial?)

What's the first non-zero term?

Q: Given $f(x)$, write the Taylor Series.

May be based on Taylor Series we know already

$\cos x$, $\sin x$, e^x, $\frac{1}{1-x}$, $\frac{1}{1+x}$, $\ln(1+x)$

$\arctan x$

d/dx

integration

substitution

$$\frac{1}{1+x^2}$$

Taylor Series (Cont'd)

Applications (less important for Ex 2)

Error Estimation:

Estimate Error of a Taylor Poly in a certain range

Taylor's Formula

$$f(x) = P_n(x) + R_n(x)$$

$$|R_n(x)| \leq M \frac{|x-a|^{n+1}}{(n+1)!}$$

where M is an upper bound on

$f^{(n+1)}(c)$ for c between x and a.

Q: Show $\Sigma *$ converges to $\underline{f(x)}$ using Taylor's Formula.

Q: Show that $\sum_{n=0}^{\infty} \frac{x^n}{n!}$ converges to e^x for all x, using Taylor's Formula.

For N, $s_N = \sum_{n=0}^{N} \frac{x^n}{n!}$ ← partial sum.

Talyor's Formula:

$$e^x = \sum_{n=0}^{N} \frac{x^n}{n!} + R_N(x)$$

if $R_N(x) \longrightarrow 0$ as $N \to \infty$

then $s_N \to e^x$ as $N \to \infty$

$$\left.0 \leq |R_N(x)| \leq \frac{\underline{e^{|x|}}\,|x|^{N+1}}{(N+1)!}\right\} \longrightarrow 0 \text{ as } N \to \infty.$$

$$\underline{\frac{d^n}{dx^n} e^x = e^x} \qquad M = e^{|x|}$$

Ex: How many terms must I sum to estimate $\sum_{n=1}^{\infty} \frac{1}{n^2}$ within 10^{-4}?

Integral Error Est:

$$\underbrace{\left| \sum_{n=1}^{\infty} a_n - \sum_{n=1}^{N-1} a_n \right|}_{\left| \sum_{n=N}^{\infty} a_n \right|} \leq \int_N^{\infty} f(x)\,dx \leq 10^{-4}$$

For what N?

$$\frac{1}{N} \leq \frac{1}{10^4}$$

$$\Leftrightarrow N \geq 10^4$$

$$\int_N^{\infty} \frac{dx}{x^2} = \lim_{b\to\infty} \int_N^b \frac{dx}{x^2}$$

$$= \lim_{b\to\infty} \left[-x^{-1} \right]_N^b$$

$$= \lim_{b\to\infty} \left[-\frac{1}{b} - \left(-\frac{1}{N}\right) \right] = \frac{1}{N}$$

A: $N = 10^4$

Binomial Series

$$(1+x)^m = \sum_{k=0}^{\infty} \binom{m}{k} x^k$$

$$\binom{m}{k} = \frac{m(m-1)\cdots(m-k+1)}{k\cdot(k-1)\cdots 3\cdot 2\cdot 1} \quad \begin{array}{l} \leftarrow k \text{ terms} \\ \leftarrow k \text{ terms} \end{array}$$

Ex: $(2+x)^m \rightsquigarrow \_\,(1+y)^m$

$$\underbrace{2^m\left(1+\frac{x}{2}\right)^m}_{\substack{(1+y)^m \\ \left(y=\frac{x}{2}\right)}} = 2^m \sum_{k=0}^{\infty} \binom{m}{k} \frac{x^k}{2^k}$$

$$(a+bx)^m = a^m \sum_{k=0}^{\infty} \binom{m}{k} \frac{b^k x^k}{a^k}$$