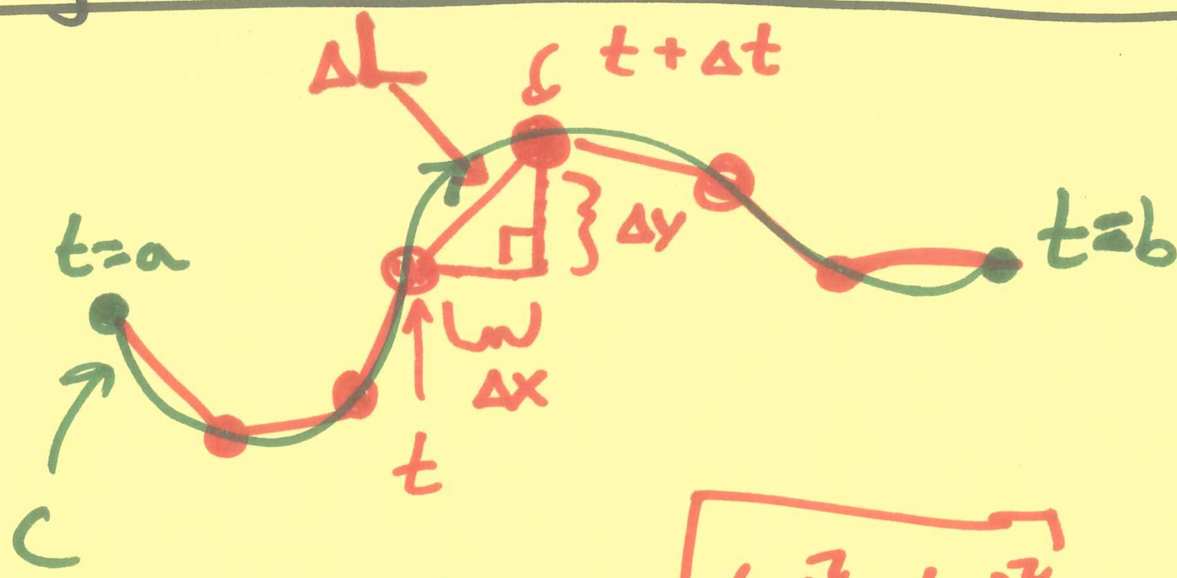
Length of a Parametric Curve



$$\Delta L = \sqrt{(\Delta x)^2 + (\Delta y)^2}$$

Length of a Parametric Curve

Let $(x, y) = (f(t), g(t))$ for $t \in [a, b]$. The length of the curve is L where

$a \le t \le b$

$$L = \int_a^b \sqrt{(f'(t))^2 + (g'(t))^2}\, dt$$

$$= \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

Length of $y = f(x)$.

$$(x,y) = (t, f(t))$$

$$L = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

$$= \int_a^b \sqrt{1 + \left(f'(t)\right)^2}\, dt$$

$$= \int_a^b \sqrt{1 + f'(x)^2}\, dx$$

Ex: Find the length of

$(x,y) = (\cos t, t + \sin t)$

for $0 \leq t \leq \pi$.

$$\frac{dx}{dt} = -\sin t, \quad \frac{dy}{dt} = 1 + \cos t$$

$$L = \int_0^{\pi} \sqrt{(-\sin t)^2 + (1+\cos t)^2}\, dt$$

$$= \int_0^{\pi} \sqrt{\underbrace{\sin^2 t + 1 + 2\cos t + \cos^2 t}_{\sin^2 t + \cos^2 t \,=\, 1}}\, dt$$

$$= \int_0^{\pi} \sqrt{2 + 2\cos t}\, dt$$

$(2 + 2\cos t = 4 \cdot (\frac{1}{2} + \frac{1}{2}\cos t)$

$$= \int_0^{\pi} \sqrt{4 \cdot \cos^2\left(\frac{t}{2}\right)}\, dt$$

$$= \int_0^{\pi} 2 \left|\cos \frac{t}{2}\right| dt$$

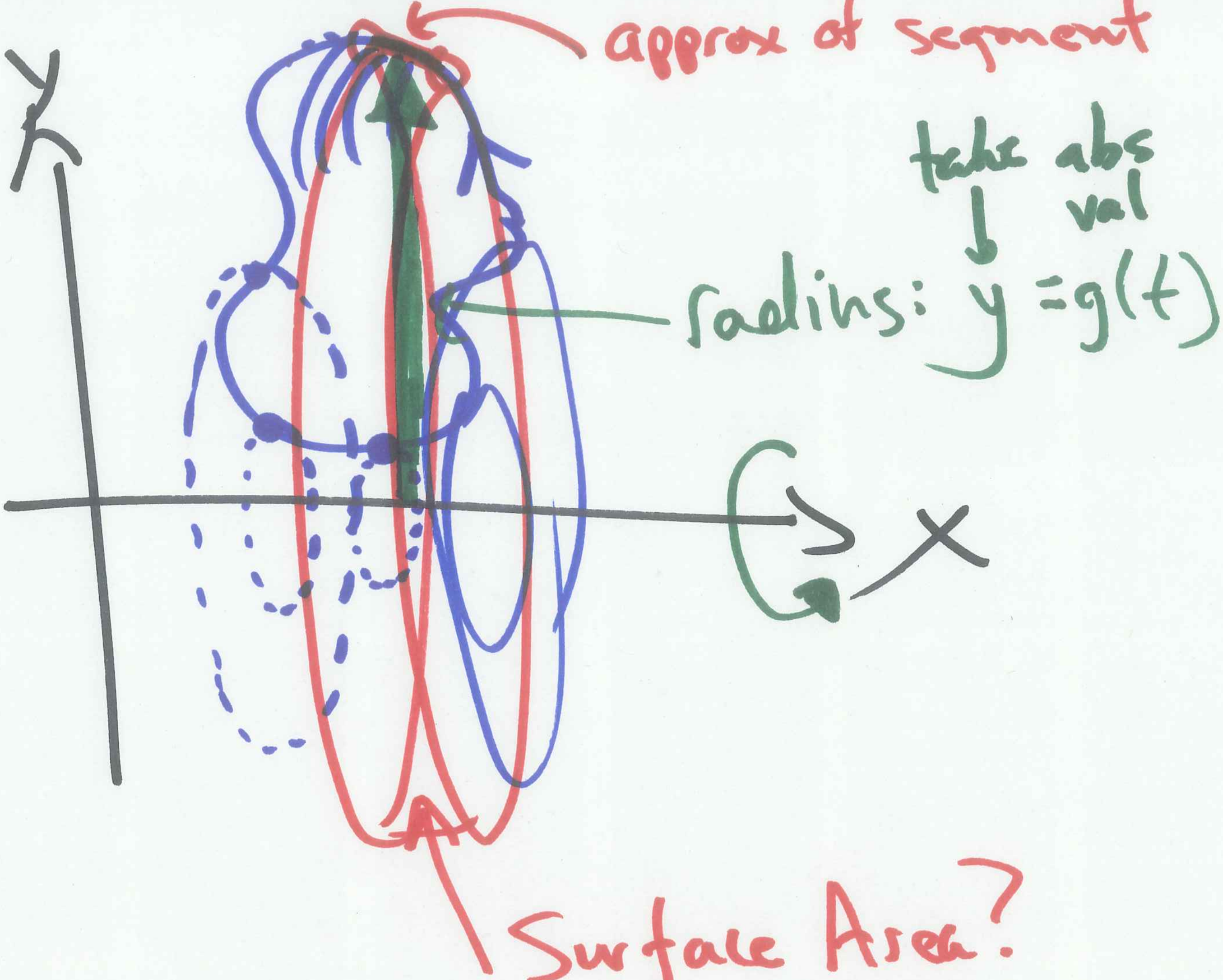
$$= \int_0^{\pi} 2 \cos\left(t/2\right) dt$$

$u = t/2$, $du = dt/2$

$$= \int_0^{\pi/2} 4 \cos u \, du$$

$$= 4 \sin u \Big]_0^{\pi/2}$$

$$= \underline{4}$$



$$dA : 2\pi y \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

$g(t)$

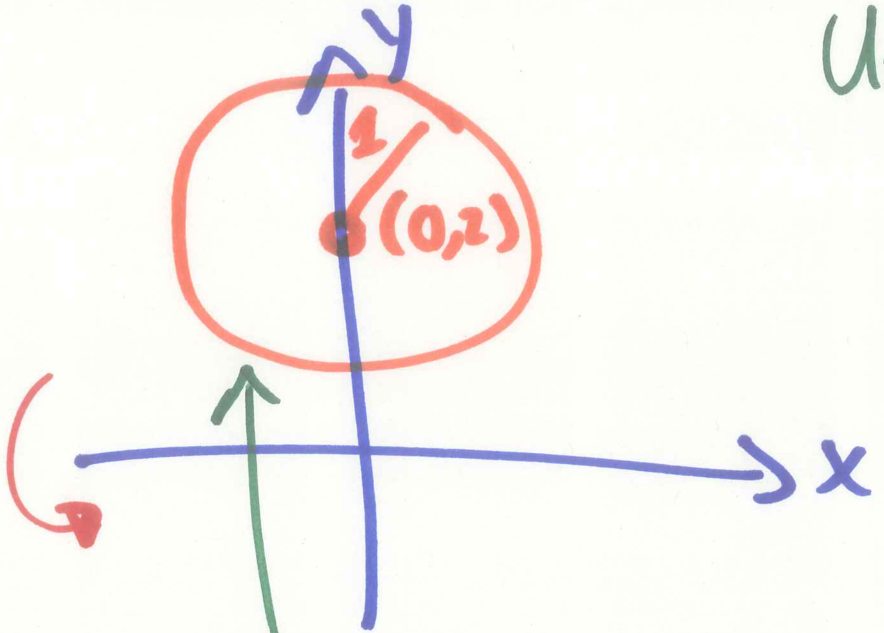
Surface Area for Surface of revolution given by a Param. Curve

If $(x,y) = (f(t), g(t))$ for $a \leq t \leq b$, then the surface area given by rotating this curve about the x-axis is

$$A = \int_a^b 2\pi y \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

$$= \int_a^b 2\pi g(t) \sqrt{f'(t)^2 + g'(t)^2}\, dt$$

(Important: $y = g(t) \geq 0$.)



Unit Circle:

$(x,y)=(\cos t, \sin t)$

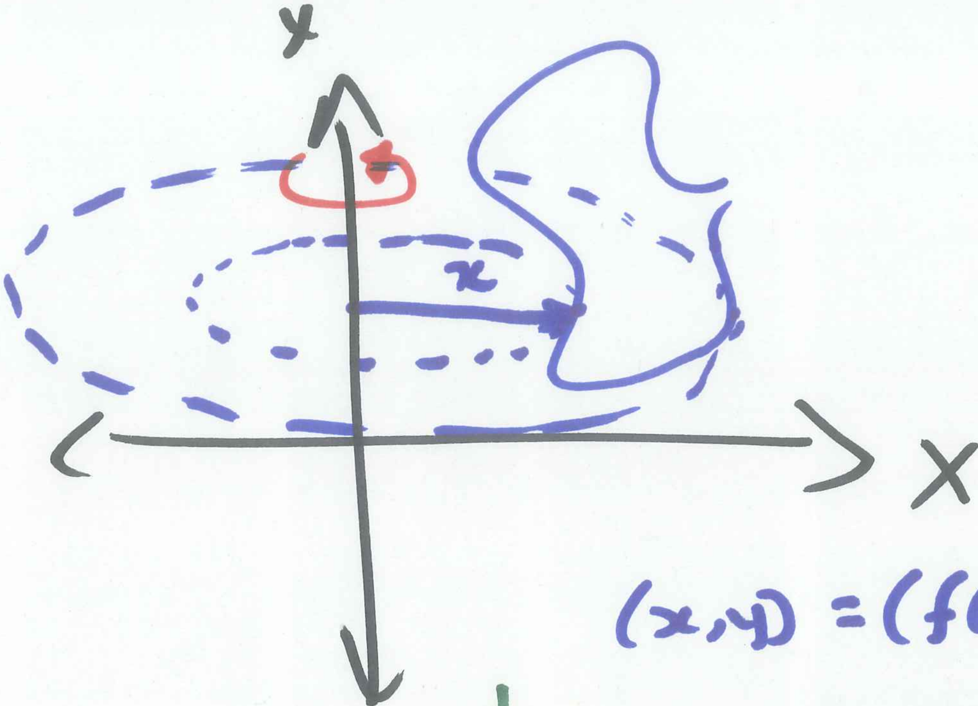
$0 \le t \le 2\pi$

$$(x,y) = (\cos t, \sin(t)+2)$$

$$A = \int_0^{2\pi} 2\pi \underbrace{(\sin t + 2)}_{y} \underbrace{\sqrt{\underbrace{(-\sin t)^2}_{\frac{dx}{dt}} + \underbrace{\cos^2 t}_{\left(\frac{dy}{dt}\right)^2}}} \, dt$$

$$= \int_0^{2\pi} 2\pi(\sin t + 2) \cdot 1 \cdot dt$$

$$= 2\pi\Big[-\cos t + 2t\Big]_0^{2\pi} = \underline{\underline{8\pi}} \text{ (sq. units)}$$



Radius: x

$(x,y) = (f(t), g(t))$

$$A = \int_a^b 2\pi x \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

$$= \int_a^b 2\pi f(t) \sqrt{f'(t)^2 + g'(t)^2}\, dt$$

$$\frac{1}{(1-x)^2} = \frac{1}{1-x} \cdot \frac{1}{1-x}$$

$$\frac{1}{1-y} = \sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + \cdots$$

$$\left(\sum_{n=0}^{\infty} x^n\right) \cdot \left(\sum_{n=0}^{\infty} x^n\right)$$

$$= (1 + x + x^2 + x^3 + \cdots)(1 + x + x^2 + x^3 + \cdots)$$

$$= 1 + 2x + 3x^2 + 4x^3 + \cdots$$

$$= \sum_{n=0}^{\infty} (n+1)\, x^n$$

$$\left(\sum a_n x^n\right)\cdot\left(\sum b_n x^n\right) = \sum c_n x^n$$

$$c_n = \underbrace{\sum_{i=0}^{n} \left(a_i \cdot b_{n-i}\right)}_{\text{involution}}$$

Ex: $\cos^2 x = \frac{1}{2} + \frac{1}{2}\cos(2x)$