1 Translating Propositions and English Sentences

Format: Given propositional variables corresponding to English sentences and a compound proposition, translate the compound proposition into an English sentence.

Problem 1.1. Let p and q be the propositions

p : I bought a lottery ticket this week.

q : I won the million dollar jackpot.

Express each of these propositions as an English sentence.

1. $\neg p$
2. $p \to q$
3. $p \wedge q$
4. $p \leftrightarrow q$
5. $\neg p \vee (p \wedge q)$

(These compound propositions may include quantifiers.)

Problem 1.2. Let $P(x, y)$ be the statement "$x > y$ and x is divisible by y." Let the domain consist of positive integers and write English sentences describing the following propositions.

a. $\exists x \forall y P(x, y)$

b. $\exists y \forall x P(x, y)$

c. $\forall x \exists y P(x, y)$

d. $\forall y \exists x P(x, y)$

Format: Given propositional variables corresponding to English sentences and a complicated English sentence, translate the English sentence into a compound proposition.

2 Constructing Truth Tables

Format: Given a compound proposition, build a truth table evaluating the proposition for all truth assignments of the propositional variables.

Method: Build a table. Add columns for each operation as to show your work.

Problem 2.1. Construct a truth table for the compound proposition $(p \leftrightarrow q) \oplus (\neg p \leftrightarrow \neg r)$.

Problem 2.2. Construct a truth table for the compound propositions $((p \to (q \to r)) \to s)$.

(Sometimes we want to determine that a compound proposition is a tautology or a contradiction, so you do the same as above, except note that every row evaluates to **T** or **F**.)

Problem 2.3. Show that $[p \wedge (p \to q)] \to q$ is a tautology using truth tables.

3 Demonstrating Logical Equivalences

Format: Given two compound propositions, demonstrate they are equivalent using logical equivalences.

Method: Start with one compound proposition, and show equivalences using the tables of logical equivalences (on cheat sheet). Use "Logical equivalence using conditionals" or "Logical equivalence using biconditionals" in order to remove/add conditionals and biconditionals so you can work with other equivalences.

Note: My strategy is as follows:

1. Use logical equivalences to remove conditionals and biconditionals ($p \leftrightarrow q \equiv (p \to q) \land (q \to p)$; $p \to q \equiv \neg p \lor q$; others).
2. Use DeMorgan's Laws and the Double-negation Law to push all negations to be directly above each propositional variable ($\neg(p \land q) \equiv \neg p \lor \neg q$; $\neg(p \lor q) \equiv \neg p \land \neg q$).
3. Use Distributive, Associative, and Commutative laws to group similar terms.
4. Use Idempotent, Identity, and Domination laws to simplify like terms.
5. Repeat above steps as necessary.

Perform the above steps starting with each side of the equation until reaching a commond mid-point, then reverse the steps from the right-hand side to create one list of logical equivalences.

Problem 3.1. Prove that $(p \to q) \lor (p \to r)$ and $p \to (q \lor r)$ are logically equivalent (without using this equivalence from the tables).

Problem 3.2. Select a logical equivalence using conditionals. Prove the equivalence without using this equivalence from the tables.

Problem 3.3. Select a logical equivalence using biconditionals. Prove the equivalence without using this equivalence from the tables.

(The compound propositions may involve quantifiers, so be sure to check the logical equivalences using quantifiers.)

Problem 3.4. Let $P(x, y)$, $Q(x, y)$, and $R(x)$ be propositional functions. Use logical equivalences to show that $\neg\forall x((\exists y(P(x, y) \to Q(x, y))) \lor R(x))$ and $\exists x(\neg R(x) \land \forall y(\neg Q(x, y) \land P(x, y)))$ are equivalent.

4 Finding Satisfying Assignments

Format: Given a compound proposition, find an assignment of the propositional variables such that the compound proposition is satisfied (i.e. true).

Method: Present the values of your variables and evaluate the compound proposition.

Problem 4.1. Find an assignment of the variables p, q, r such that the proposition

$$(p \lor \neg q) \land (p \lor q) \land (q \lor r) \land (q \lor \neg r) \land (r \lor \neg p) \land (r \lor p)$$

5 Rules of Inference

Format: Given a set of premises, *use rules of inference* to prove a conclusion.

Method: Use tabular "step/reason" proof to list true propositions and how they are built from premises and previous conclusions until final conclusion is determined. Both Rules of Inference and Logical Equivalences can be used (and will be on cheat sheet).

Problem 5.1. Use rules of inference to show that if $p \wedge q$, $r \vee s$, and $p \to \neg r$, then s is true.

Problem 5.2. Use rules of inference to show that if $(p \to q) \wedge (q \to p)$, $t \vee q$, $t \vee p$, and $(p \wedge q) \to t$, then t is true.

Problem 5.3. Use rules of inference to show that if $(p \wedge q) \vee r$, $(p \wedge \neg q) \vee s$, $(\neg p \wedge q) \vee t$, and $(\neg p \wedge \neg q) \vee u$, then $r \vee s \vee t \vee u$ is true.

Format: Given a rule of inference, use truth tables to demonstrated that the rule is correct.

Method: Build a truth table constructed using all propositional variables in the rule. Cross out the rows where any of the premises are false. Determine that the conclusion is true for all remaining rows.

Problem 5.4. Use truth tables to demonstrate that Resolution is a correct rule of inference.

Problem 5.5. Use truth tables to demonstrate that Conditional Resolution is a correct rule of inference.

6 Knights, Knaves, and Others

Format: Given a set of possible types of people that say propositions, determine what possible types the people can be.

Method: Use truth tables or proof by case analysis.

Problem 6.1. On the island of Flopi, there are three types of people: Knights, Knaves, and Floppers. All inhabitants know which type the others are, but they are otherwise indistinguishable. Knights always tell the truth. Knaves always lie. Floppers always choose to lie or tell the truth by doing the opposite of the previous speaker (i.e. if someone just spoke a lie, the flopper will tell the truth; if someone just spoke a truth, the flopper will lie). While on your vacation, you come across three inhabitants, A, B, and C. They say the following, in order:

> A says, "We are all knights."
> B says, "C is a knight."
> C says, "A is a knave."
> A says, "C lied."

Determine all possibilities of A, B, and C being Knights, Knaves, or Floppers (not all need to be distinct).

Problem 6.2. In the country of Togliristan (where Knights, Knaves, and Togglers live), Togglers will alternate between telling the truth and lying (no matter what other people say). You meet two people, A and B. They say, in order:

> A : B is a Knave.
>
> B : A is a Knave.
>
> A : B is a Knight.
>
> B : A is a Toggler.

Determine what types of people A and B are.

7 Basic Proofs

Format: Give a proof of a statement. May require a specific type of proof (direct, contrapositive, contradiction).

Method: Determine if proof is of type "$p \to q$" or just "p" and determine how to organize a direct proof (Assume p is true. Use that information to demonstrate q is true), contrapositive (Assume $\neg q$ is true. Use that information to demonstrate $\neg p$ is true), or contradiction (Assume $p \wedge \neg q$ is true and find a contradiction; Assume $\neg p$ is true and find a contradiction.)

Tip: It can be helpful to carefully write out what you mean by p, q, $\neg p$, or $\neg q$, when you are trying to demonstrate your proof technique. Also, if you are given a choice of proof method, then announce your method at the beginning of your proof.

Problem 7.1. Give a direct proof of the following statement: If x and y are rational numbers, then $x + y$ and xy are rational numbers.

Problem 7.2. Give a proof of the following statement: If x is a rational number and y is irrational, then $x + y$ and xy are irrational.

(You may need to adapt an existing proof in order to give a proof of the statement.)

Problem 7.3. Prove that $\sqrt{3}$ is irrational.

Problem 7.4. Prove that $\sqrt{7}$ is irrational.

Problem 7.5. Prove that $\sqrt{35}$ is irrational.

Problem 7.6. Prove that $\sqrt{12}$ is irrational.

8 Tiling Problems

Format: Given a description of a chessboard (either an $m \times n$ chessboard, or one that has had some squares removed), demonstrate that a tiling exists or does not exist using some set of polyominoes.

Method: For existence of a tiling: demonstrate the tiling (usually by tiling a small part of the board and then replicating that tiling across the board). For non-existence of a tiling, prove something along the lines of "If C has a tiling, then C has property X." Then show that C does not have property X.

Problem 8.1. Prove that if n is even, then the $m \times n$ chessboard has a domino tiling.

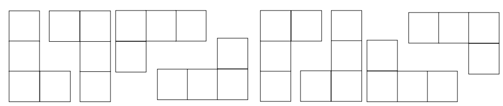
Problem 8.2. Prove that if n and m are odd, then the $m \times n$ chessboard does not have a domino tiling.

Problem 8.3. Let n be an even number. Prove that if we remove the top-left and bottom-right squares from the $n \times n$ chessboard, the resulting board does not have a domino tiling.

Problem 8.4. Let n be a multiple of 3. Prove that the $m \times n$ chessboard has a tiling using straight triominoes.

Problem 8.5. Let n be a number not divisible by 3. Prove that the $n \times n$ chessboard does not have a tiling using straight triominoes.

Problem 8.6. The *L-shaped Tetris piece* (or *tetromino*, see the Wikipedia page) consists of four squares: three of which are in a line and a fourth attached to one end of that line. See the figure below for all of the arrangements of the L-shaped Tetris piece (or L-piece).



Adapt the proofs from the book and class about domino tilings to prove the following:

a. If m is an even number and n is a multiple of 4, then the $m \times n$ chessboard can be tiled using L-pieces.

b. If a chessboard has a tilling using L-pieces, then the chessboard has a domino tiling.

c. If m and n are odd numbers, then the $m \times n$ chessboard cannot be tiled using L-pieces.

d. Show that if k is even, then the $3 \times 4k$ chessboard can be tiled using L-pieces.

Problem 8.7. Let the *T-piece* be the tetromino (4 squares) that covers positions $(1, 1)$, $(1, 2)$, $(1, 3)$, and $(2, 2)$.

a. Prove that if n and m are multiples of 4, then the $m \times n$ chessboard can be tiled using T-pieces.

b. Prove that the $2 \times n$ chessboard cannot be tiled using T-pieces.

c. Prove that the $2 \times n$ chessboard with the top-left and bottom-right squares removed can be tiled using T-pieces.

d. Prove that the $3 \times n$ chessboard cannot be tiled using T-pieces.

9 Set Definitions

Problem 9.1. Let $\mathcal{U} = \{1, 2, 3, 4, 5, 6, 7\}$, $A = \{1, 3, 5, 7\}$, $B = \{4, 5, 6, 7\}$. Determine the following sets:

$$\overline{A}, \quad A \cap B, \quad A \cup B, \quad A \setminus B, \quad A \triangle B.$$

Problem 9.2. Consider our definitions of $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$, and $\mathbb{C}$. Recall that $A \subseteq B$ means "A is a subset of B" and $A \not\subseteq B$ means "A is not a subset of B."

Prove that (a) $\mathbb{Z} \subseteq \mathbb{Q}$, (b) $\mathbb{Q} \not\subseteq \mathbb{Z}$, (c) $\mathbb{R} \not\subseteq \mathbb{Q}$, (d) $\mathbb{R} \subseteq \mathbb{C}$, and (e) $\mathbb{C} \not\subseteq \mathbb{R}$.

Problem 9.3. Recall the definition of the cartesian product $A \times B$.

Prove that (a) If $A \subseteq B$ then $A \times A \subseteq B \times B$. (b) $\mathbb{Z} \times \mathbb{Q} \subseteq \mathbb{Q} \times \mathbb{R}$.

10 Set Containment or Equality

Format: Given some sets, determine if a set built from set operations is a subset of or is equal to another set.

Method: To demonstrate $A \subseteq B$, you must say that if $a \in A$ then $a \in B$. To demonstrate $A = B$ you must show $A \subseteq B$ and $B \subseteq A$.

Problem 10.1. Let A, B, and C be sets. Prove that $(A \triangle B) \triangle C = (A \setminus (B \cup C)) \cup (B \setminus (A \cup C)) \cup (C \setminus (A \cup B))$.

(Note: $\mathcal{P}(A)$ is the *power set* of A and consists of *subsets of* A. That is, $S \in \mathcal{P}(A)$ if and only if $S \subseteq A$.)

Problem 10.2. Prove that if $A \subseteq B$, then $\mathcal{P}(A) \subseteq \mathcal{P}(B)$.

Problem 10.3. Let A and B be sets. Prove that $\mathcal{P}(A) \cap \mathcal{P}(B) = \mathcal{P}(A \cap B)$.

Problem 10.4. Let A and B be sets. Prove that $\mathcal{P}(A) \cup \mathcal{P}(B) \subseteq \mathcal{P}(A \cup B)$.

Problem 10.5. Let A and B be sets. Prove that if $A \not\subseteq B$ and $B \not\subseteq A$ then $\mathcal{P}(A) \cup \mathcal{P}(B) \neq \mathcal{P}(A \cup B)$.

11 Set Equality by Set Identities

Format: Use definitions of set operations and set identities to prove an equality of sets.

Method: Start with the left-hand side of the equality, use definitions and set identities to step-by-step transform the set into the right-hand side.

Note: This is very close to the method and format for proving logical equivalences (they are actually the same thing, after performing a translation between set operations and logical operations). My strategy is as follows:

1. Use set definitions to remove any symmetric differences ($A \triangle B = (A \setminus B) \cup (B \setminus A)$) and set differences ($A \setminus B = A \cap \overline{B}$).
2. Use DeMorgan's Laws and the Complementation Law to push all complements to be directly above each set ($\overline{A \cup B} = \overline{A} \cap \overline{B}$; $\overline{A \cap B} = \overline{A} \cup \overline{B}$).
3. Use Distributive, Associative, and Commutative laws to group similar terms.
4. Use Idempotent, Identity, and Domination laws to simplify like terms.
5. Repeat above steps as necessary.

Perform the above steps starting with each side of the equation until reaching a commond mid-point, then reverse the steps from the right-hand side to create one list of set equalities.

Problem 11.1. Let A, B, and C be subsets of a universe $\mathcal{U}$. Use definitions of set operations and set identities to prove the following equality of sets:

$$((B \cap A) \cup (B \cap C)) \setminus (A \cap B \cap C) = (B \cap (A \triangle C))$$

[Hint: Show both sides are equal to the set $((B \cap A) \setminus C) \cup ((B \cap C) \setminus A)$.]

Problem 11.2. Let A, B, and C be subsets of a universe $\mathcal{U}$. Use definitions of set operations and set identities to prove the following equality of sets:

$$(A \triangle B) \triangle C = (A \setminus (B \cup C)) \cup (B \setminus (A \cup C)) \cup (C \setminus (A \cup B)).$$

Problem 11.3. Use definitions of set operations and set identities to prove that the symmetric difference is associative. That is

$$(A \triangle B) \triangle C = A \triangle (B \triangle C).$$

12 Set Sizes

Format: Given a set built from others using set operations, prove the size of the set equals the given formula.

Method: Count how many elements are in each part of the set.

Problem 12.1. Let A and B be sets. Prove that $|A \cup B| = |A| + |B| - |A \cap B|$, using the following steps:

1. Prove that if E and F are disjoint sets (i.e. $E \cap F = \varnothing$) then $|E \cup F| = |E| + |F|$.
2. Prove that $|A \cup B| = |A| + |B \setminus A|$.
3. Prove that $|B \setminus A| = |B| - |A \cap B|$.
4. Conclude that $|A \cup B| = |A| + |B| - |A \cap B|$.

13 Types of Functions

Format: Given a function, determine if it is injective, surjective, both, or neither.

Method: Let $f : A \to B$ be a function.

- To show f is injective: Prove "$\forall x, y \in A, (f(x) = f(y) \to x = y)$."
- To show f is *not* injective: Prove "$\exists x, y \in A, (f(x) = f(y) \land x \neq y)$."
- To show f is surjective: Prove "$\forall y \in B, \exists x \in A, f(x) = y$."
- To show f is *not* surjective: Prove $\exists y \in B, \forall x \in A, f(x) \neq y$."

Note: Be sure to know how to build functions from other functions, including $f_1 + f_2$, $f_1 f_2$, and $f_1 \circ f_2$.

Problem 13.1. Let $A = \{a, b, c\}$, $B = \{1, 2, 3, 4\}$, and $C = \{\pi, \phi, i\}$. Define functions $f : A \to B$ and $g : B \to C$ as

$$f(x) = \begin{cases} 2 & x = a \\ 3 & x = b \\ 4 & x = c \end{cases} \qquad g(x) = \begin{cases} \pi & x = 1 \\ \phi & x = 2 \\ i & x = 3 \\ \pi & x = 4 \end{cases}$$

Consider each of the functions f, g, $g \circ f$ and determine if they are injective, surjective, or both.

Problem 13.2. Consider the following function $f : \mathbb{N} \to \mathbb{Z}$:

$$f(n) = (-1)^n \left(\frac{n}{2} + \frac{1}{4}\right) - \frac{1}{4}.$$

1. Write out the elements $f(0)$, $f(1)$, $f(2)$, $f(3)$, $f(4)$, $f(5)$.
2. Prove that f is injective.
3. Prove that f is surjective.
4. Describe a function $g : \mathbb{Z} \to \mathbb{Q}$ that is surjective (and prove it is surjective).

Problem 13.3. Let $f : A \to B$ and $g : B \to C$ be functions. Prove that if f and g are bijections, then $g \circ f$ is a bijection.