Type your answers to the following questions and submit a PDF file to Blackboard. One page per problem.

Problem 1. [10pts] Let $M = (S, T, s_0)$ be the state machine where $S = \mathbb{N} \times \mathbb{N}$ and the transitions in T are given as $(m, n) \to (m-1, n+1)$ if $m \geq 1$, and $(m, n) \to (m, n-1)$ if $n \geq 1$. Define a function $f : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ where f is decreasing with respect to M and use the Monotonicity Principle to prove that if the initial state is (m, n), then the machine will halt in at most $f(m, n)$ transitions. (Bonus 1pt for defining $f(m, n)$ to be *optimal*, predicting exactly the maximum number of steps from (m, n) to a halting state.) [Hint: Draw the points (m, n) in the plane and draw lines between points for possible transitions.]

Problem 2. [10pts] Let $\Sigma = \{a^+, a^-, b^+, b^-\}$. Let $M = (S, T, s_0)$ be the state machine where $S = \Sigma^*$ and the transitions available in T from a string $x_1 \dots x_n \in \Sigma^*$ are as follows:

- If there exists $i \in \{1, \dots, n-1\}$ such that $x_i = a^+$ and $x_{i+1} = a^-$, then $x_1 \dots x_n \longrightarrow x_1 \dots x_{i-1}x_{i+2} \dots x_n$.
- If there exists $i \in \{1, \dots, n-1\}$ such that $x_i = a^-$ and $x_{i+1} = a^+$, then $x_1 \dots x_n \longrightarrow x_1 \dots x_{i-1}x_{i+2} \dots x_n$.
- If there exists $i \in \{1, \dots, n-1\}$ such that $x_i = b^+$ and $x_{i+1} = b^-$, then $x_1 \dots x_n \longrightarrow x_1 \dots x_{i-1}x_{i+2} \dots x_n$.
- If there exists $i \in \{1, \dots, n-1\}$ such that $x_i = b^-$ and $x_{i+1} = b^+$, then $x_1 \dots x_n \longrightarrow x_1 \dots x_{i-1}x_{i+2} \dots x_n$.

a. [5pts] Describe all of the possible state sequences when starting at the string $b^+a^+b^+b^-a^-b^+a^+b^-b^+$.

b. [5pts] For certain states in Σ^*, there may be more than one outgoing transition. Prove that for every string $\mathbf{x} \in \Sigma^*$, there is a unique halting state reachable from $\mathbf{x}$. [Hint: Use induction on the length of $\mathbf{x}$, and consider the following strengthened statement: For every state $\mathbf{x}$ with outgoing transitions $\mathbf{x} \to \mathbf{y}^{(i)}$ for $i \in \{1, \dots, k\}$, all states $\mathbf{y}^{(1)}, \dots, \mathbf{y}^{(k)}$ halt in the same state, $\mathbf{z}$.]

Problem 3. [10pts] Let $M = (\mathbb{Q}, T, s_0)$ be a state machine on the rational numbers with transitions $\frac{p}{q} \to \frac{p}{q} + \frac{1}{pq} = \frac{p^2+1}{pq}$, when $p > 0$ and $q > 0$.

a. [5pts] Let $P_d(\frac{p}{q})$ be the property "$d \leq \frac{p}{q} < d + 1$." Prove that when d is an integer with $d \geq 2$, P_d is a reversible invariant for the state machine M. [Hint: If $P_d(\frac{p}{q})$ is true, then $p = dq + r$ where r is an integer and $0 \leq r < q$.]

b. [5pts] Use part (a) and the Reversibility Principle to say that if $\frac{p}{q}$ is the initial state and $\frac{i}{j}$ is a reachable state, then $\lfloor \frac{p}{q} \rfloor = \lfloor \frac{i}{j} \rfloor$. [You can get full points for this part if you assume (a), even if you have not proven (a) correctly.]

c. [Bonus 1pt] Find a fraction $\frac{p}{q}$ such that $P_1(\frac{p}{q})$ is true but $P_1(\frac{p^2+1}{pq})$ is false.

Problem 4. [10pts] Define a set $S \subseteq \mathbb{R}$ recursively using the Recursive Step "If $x \in S$, then $(x-1)(x+1) \in S$." Consider the following bases.

a. [3pts] Prove that if the basis step is "$0 \in S$" then $S = \{0, -1\}$.

b. [3pts] Prove that if the basis step is "$1 \in S$" then $S = \{1, 0, -1\}$.

c. [4pts] Prove that if the basis step is "$2 \in S$" then S is an infinite set.

Problem 5. [10pts] Define a set $S \subseteq \mathbb{R}$ recursively by (Basis Step) $\frac{1}{1} \in S$, and (Recursive Step) If $x \in S$, then $2x \in S$ and $\frac{x}{3} \in S$. Prove that $S = \{\frac{2^i}{3^j} : i, j \in \mathbb{N}\}$.