Type your answers to the following questions and submit a PDF file to Blackboard. One page per problem.

Problem 1. [10pts] Let $n_1, \ldots, n_t$ be positive integers. Prove that if $1 + \sum_{i=1}^{t} n_i$ objects are placed into t boxes, then for some i, the ith box contains at least $n_i + 1$ objects.

Proof. We prove by the contrapositive. Suppose some number of objects are placed into t boxes such that the ith box contains at most n_i objects. Then the number of objects is at most $\sum_{i=1}^{t} n_i$. Thus, we did not place $1 + \sum_{i=1}^{t} n_i$ objects. $\square$

Problem 2. [15pts] Recall the definition of the Ramsey number $r(k, \ell)$.

a. [5pts] Prove that $r(k, 2) = k$.

Proof. Let $c : E_k \to \{R, B\}$ be a 2-coloring. If any pair $\{i, j\} \in E_k$ receives color B then there exists a B-colored clique of size 2. Otherwise, all pairs $\{i, j\} \in E_k$ receive color R and there exists an R-colored clique of size k. Thus, $r(k, 2) \leq k$.

To show $r(k, 2) > k - 1$, use the 2-coloring $c : E_{k-1} \to \{R, B\}$ where every pair $\{i, j\} \in E_{k-1}$ receives the color R. Thus there is no B-colored clique of size 2 and there is no clique of size k at all. □

b. [10pts] Use problem 1, part (a), and the fact that $r(3, 3) = 6$ to prove that $r(3, 4) \leq 10$.

Proof. Let $c : E_{10} \to \{R, B\}$ be a 2-coloring. We will show that c contains either an R-colored clique of size 3 or a B-colored clique of size 4.

Consider the pairs $\{i, 10\}$ for $1 \leq i \leq 9$. There are 9 pairs that receive two colors. Let $n_1 = 3$ and $n_2 = 5$. Since $9 = 1 + (3 + 5)$, then by Problem 1 either there are $n_1 + 1 = 4$ pairs $\{i, 10\}$ that are colored B or there are $n_2 + 1 = 6$ pairs $\{i, 10\}$ that are colored R.

If there are 4 pairs $\{i, 10\}$ colored B, without loss of generality we can let the pairs $\{i, 10\}$ with $1 \leq i \leq 4$ be the pairs colored B. Since $r(2, 4) = r(4, 2) = 4$ by part (a), the 2-coloring c on pairs $\{i, j\}$ with $1 \leq i < j \leq 4$ either has a B-colored edge (completing a B-colored clique of size 3 with the vertex 10) or has an R-colored clique of size 4. Thus, c contains a B-colored clique of size 3 or an R-colored clique of size 4.

If there are 6 pairs $\{i, 10\}$ colored R, without loss of generality we can let the pairs $\{i, 10\}$ with $1 \leq i \leq 6$ be the pairs colored R. Since $r(3, 3) = 6$, the 2-coloring c on pairs $\{i, j\}$ with $1 \leq i < j \leq 6$ either has a B-colored clique of size 3 or has an R-colored clique of size 3 (completing an R-colored clique of size 4 with the vertex 10). Thus, c contains a B-colored clique of size 3 or an R-colored clique of size 4. □

Problem 3. [10pts] Let k and n be integers with $0 \leq k \leq n$. Give a formula for the coefficient of x^{2k} in the polynomial $(x + \frac{1}{x})^{2n}$. Simplify the formula as much as possible.

Proof. By the binomial theorem with $x = x$ and $y = \frac{1}{x}$, we have

$$\left(x + \frac{1}{x}\right)^{2n} = (x+y)^{2n} = \sum_{i=0}^{2n} \binom{2n}{i} x^i y^{2n-i} = \sum_{i=0}^{2} n \binom{2n}{i} x^i x^{-(2n-i)} = \sum_{i=0}^{2n} \binom{2n}{i} x^{2i-2n}.$$

The monomial x^{2k} appears when $2k = 2i - 2n$, hence when $k = i - n$ and $i = n + k$. Thus, the coefficient of x^{2k} is given by the binomial coefficient $\binom{2n}{n+k}$. $\square$

Problem 4. [10pts] Give a combinatorial proof that $\sum_{k=0}^{n} k\binom{n}{k}^2 = n\binom{2n-1}{n-1}$. [Hint: Select a committee of size n from a group of n mathematicians and n computer scientists and select a chair for the committee that is a mathematician.]

Proof. We will use double-counting to select a committee of size n from a group of n mathematicians and n computer scientists and select a chair for the committee that is a mathematician.

We could select a mathematician as the chair in n ways. There are $2n-1$ people remaining to fill the remaining $n-1$ positions in the committee, so there are $\binom{2n-1}{n-1}$ ways to complete the committee. Thus, there are $n\binom{2n-1}{n-1}$ ways to select this committee.

We could also first select the number of mathematicians in the committee. Let k be the number of mathematicians to be on the committee. Then there are $\binom{n}{k}$ ways to select the mathematicians. There are $\binom{n}{n-k}$ ways to select the computer scientists on the committee. Recall that $\binom{n}{n-k} = \binom{n}{k}$. Finally, there are k mathematicians in the committee for us to select the chair. Hence, there are $\sum k = 1^n k\binom{n}{k}^2$ possible selections. Since $0\binom{n}{0}^2 = 0$, we can add the $k = 0$ term to our sum, resulting in $\sum_{k=0}^{n} k\binom{n}{k}^2$ possible committees.

Since these two expressions are counting the same thing, they must be equal. $\square$