1 Monday, February 2

1.1 Rosen 2.1 – Sets

Reading: Rosen 2.1. LLM 4.1. Ducks 2.1, 2.2.

Def: A *set* is an unordered collection of objects, called *elements* or *members* of the set. A set is said to *contain* its elements. We write $a \in A$ to denote that a is an element of the set A. We write $a \notin A$ to denote that a is not an element of the set A.

A set can be described by listing its entries between curly braces: $A = \{a, b, c, d\}$ (the *roster method*).

A set can be defined using propositional logic, where the set contains all elements that satisfy the propositional function: $A = \{x : P(x) \text{ is true }\}$. (Some use colon, some use vertical bar. Both are accepted, but I like the colon.)

Ex:

Def: $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Z}^+$, $\mathbb{Q}$, $\mathbb{R}$, $\mathbb{R}^+$, $\mathbb{C}$.

Def: Intervals $[a, b]$, $[a, b)$, $(a, b]$, (a, b).

Ex: $\{\mathbb{N}, \mathbb{Q}, \mathbb{Z}, \mathbb{R}, \mathbb{C}\}$

Def: Set are *equal* if they contain the same elements.

Def: Empty set $\varnothing$, *singleton set*.

Note: We say elements are "objects" without defining "object." This is on purpose: a formal definition is outside the scope of this class. In the current definition, (and almost all definitions of set theory) we can find *paradoxes*.

1.2 Subsets

Def: Let A and B be sets. A is a *subset* of B, denoted $A \subseteq B$, if every element of A is an element of B. (If A and B are also not equal, then we use $A \subset B$.)

Ex: $A = \{1, 2, 3\}$, $B = \{1, 2, 3, 4, 5\}$, $C = \{1, 3, 5\}$.

To show that A is a subset of B, you demonstrate $\forall x, (x \in A) \to (x \in B)$.

To show that A is *not* a subset of B, you demonstrate $\exists x, (x \in A) \land (x \notin B)$.

Thm: For every set S, $\varnothing \subseteq S$ and $S \subseteq S$.

To show that A and B are equal, you demonstrate $A \subseteq B$ and $B \subseteq A$.

Def: Let S be a set. If there are exactly n distinct elements of S, then S is a *finite set* and n is the *cardinality* of S, denoted $|S|$. (If S has an infinite number of elements, then we say $|S| = \infty$.)

Def: If S is a set, then the *power set* of S, denoted $\mathcal{P}(S)$ or 2^S, is the set of subsets of S.

1.3 Cartesian Products

Def: An *ordered n-tuple* $(a_1, a_2, \ldots, a_n)$ is the ordered collection that has a_1 as its first element, a_2 as its second element, and so on until a_n the nth element.

2-tuples are called *ordered pairs*.

Def: If A and B are sets, then the *cartesian product* of A and B, denoted $A \times B$, is the set of ordered pairs

(a, b) where the first element a is an element of A and the second element b is an element of B.

$$A \times B = \{(a, b) : a \in A \wedge b \in B\}.$$

Def: If $A_1, \ldots, A_n$ are sets, then the *cartesian product* of $A_1, \ldots, A_n$, denoted $A_1 \times A_2 \times \cdots \times A_n$ or $\prod_{i=1}^{n} A_i$, is the set of n-tuples $(a_1, a_2, \ldots, a_n)$ where for all i with $1 \leq i \leq n$, the ith element a_i is an element of A_i.

$$\prod_{i=1}^{n} A_i = A_1 \times A_2 \times \cdot \times A_n = \{(a_1, a_2, \ldots, a_n) : \forall i \in \{1, \ldots, n\} a_i \in A_i\}.$$

1.3.1 Suggested Homework

Rosen 2.1: 1,3–8,9–11, 14–17, 18–24, 25*, 26–30, 35–37, 46*, 47*.

2 Wednesday, February 4

2.1 Rosen 2.2 – Set Operations

Reading: Rosen 2.2, LLM 4.1, 4.4, Ducks 2.2.4.

Def: *union* $A \cup B$, *intersection* $A \cap B$, *disjoint* ($A \cap B = \varnothing$), *difference* ($A - B$ or $A \setminus B$) also called *complement of* B *with respect to* A, *complement* ($\overline{A} = U \setminus A$) [Redefine difference as $A \setminus B = A \cap \overline{B}$], *symmetric difference* ($A \oplus B = A \triangle B = (A \setminus B) \cup (B \setminus A)$)

Thm: Let A and B be subsets of U. $A \triangle B = (A \cup B) \setminus (A \cap B)$.

Proof. Let A and B be subsets of U. The set $A \triangle B$ consists of elements $a \in U$ such that ($a \in A$ and $a \notin B$) or ($a \in B$ and $a \notin A$). The set $(A \cup B) \setminus (A \cap B)$ consists of elements $a \in U$ such that ($a \in A$ or $a \in B$) and not ($a \in A$ and $a \in B$). To show equality, we must show $A \triangle B \subseteq (A \cup B) \setminus (A \cap B)$ and $(A \cup B) \setminus (A \cap B) \subseteq A \triangle B$.

($A \triangle B \subseteq (A \cup B) \setminus (A \cap B)$) Let $a \in A \triangle B$. Since ($a \in A$ and $a \notin B$) or ($a \in B$ and $a \notin A$), we consider two cases. If $a \in A$ and $a \notin B$, then a is in $A \cup B$ and a is not in $A \cap B$. Therefore, a is in $(A \cup B) \setminus (A \cap B)$. If $a \in B$ and $a \notin A$, then a is in $A \cup B$ and a is not in $A \cap B$. Therefore, a is in $(A \cup B) \setminus (A \cap B)$.

($(A \cup B) \setminus (A \cap B) \subseteq A \triangle B$) Let $a \in (A \cup B) \setminus (A \cap B)$. Since $a \in (A \cup B)$, we have that $a \in A$ or $a \in B$. If $a \in A$, then since $a \notin (A \cap B)$, we have $a \notin B$. Therefore, $a \in A \triangle B$. If $a \in B$, then since $a \notin (A \cap B)$, we have $a \notin A$. Therefore, $a \in A \triangle B$. □

This would be easier if we have some helpful tools. We will prove it again later.

Note: Suppose the universe U is a finite set. Then list the elements of U as $a_1, a_2, \ldots, a_n$ where $n = |U|$. For each set $A \subseteq U$, we can associate A with a binary string $\mathbf{x}_A = (x_1, \ldots, x_n)$ where $x_i = 1$ if and only if $a_i \in A$. So, this bit string encodes the truth values of the n propositions $p_i =$ "$a_i \in A$".

	And/Intersection	**Or/Union**	**Xor/Symmetric Difference**
Sets	$A \cap B$	$A \cup B$	$A \triangle B$
Logic	$(a \in A \cap B) \leftrightarrow (a \in A \wedge a \in B)$	$(a \in A \cup B) \leftrightarrow (a \in A \vee a \in B)$	$(a \in A \triangle B) \leftrightarrow (a \in A \oplus a \in B)$
Bit Strings	$\mathbf{x}_{A \cap B} = \mathbf{x}_A \wedge \mathbf{x}_B$	$\mathbf{x}_{A \cup B} = \mathbf{x}_A \vee \mathbf{x}_B$	$\mathbf{x}_{A \triangle B} = \mathbf{x}_A \oplus \mathbf{x}_B$

See Table **??** for a list of set identities.

Thm: Let A and B be subsets of U. $A \triangle B = (A \cup B) \setminus (A \cap B)$.

Proof. We will use set identities.

TABLE: **Logical Equivalences and Set Identities**		
Logical Equivalence	*Set Identity*	*Name*
$p \wedge \mathbf{T} \equiv p$ $p \vee \mathbf{F} \equiv p$	$A \cap \mathcal{U} = A$ $A \cup \varnothing = A$	Identity laws
$p \vee \mathbf{T} \equiv \mathbf{T}$ $p \wedge \mathbf{F} \equiv \mathbf{F}$	$A \cup \mathcal{U} = \mathcal{U}$ $A \cap \varnothing = \varnothing$	Domination laws
$p \vee p \equiv p$ $p \wedge p \equiv p$	$A \cup A = A$ $A \cap A = A$	Idempotent laws
$\neg(\neg p) \equiv p$	$\overline{(\overline{A})}$	Double negation law / Complementation law
$p \vee q \equiv q \vee p$ $p \wedge q \equiv q \wedge p$	$A \cup B = B \cup A$ $A \cap B = B \cap A$	Commutative laws
$(p \vee q) \vee r \equiv p \vee (q \vee r)$ $(p \wedge q) \wedge r \equiv p \wedge (q \wedge r)$	$(A \cup B) \cup C = A \cup (B \cup C)$ $(A \cap B) \cap C = A \cap (B \cap C)$	Associative laws
$p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$ $p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$	$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$ $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$	Distributive laws
$\neg(p \wedge q) \equiv \neg p \vee \neg q$ $\neg(p \vee q) \equiv \neg p \wedge \neg q$	$\overline{A \cap B} = \overline{A} \cup \overline{B}$ $\overline{A \cup B} = \overline{A} \cap \overline{B}$	De Morgan's laws
$p \vee (p \wedge q) \equiv p$ $p \wedge (p \vee q) \equiv p$	$A \cup (A \cap B) = A$ $A \cap (A \cup B) = A$	Absorption laws
$p \vee \neg p \equiv \mathbf{T}$ $p \wedge \neg p \equiv \mathbf{F}$	$A \cup \overline{A} = \mathcal{U}$ $A \cap \overline{A} = \varnothing$	Negation laws / Complement laws

Table 1: Set Identities.

$$
\begin{aligned}
A \triangle B &= (A \setminus B) \cup (B \setminus A) && \text{Definition of Symmetric Difference} \\
&= (A \cap \overline{B}) \cup (B \cap \overline{A}) && \text{Definition of Set Subtraction} \\
&= ((A \cap \overline{B}) \cup B) \cap ((A \cap \overline{B}) \cup \overline{A}) && \text{Distributive Law} \\
&= ((A \cup B) \cap (\overline{B} \cup B)) \cap ((A \cap \overline{A}) \cup (\overline{B} \cap \overline{A})) && \text{Distributive Law} \\
&= ((A \cup B) \cap U) \cap ((A \cup \overline{A}) \cap (\overline{B} \cup \overline{A})) && \text{Complement Law} \\
&= ((A \cup B) \cap U) \cap (U \cap (\overline{B} \cup \overline{A})) && \text{Complement Law} \\
&= (A \cup B) \cap (U \cap (\overline{B} \cup \overline{A})) && \text{Identity Law} \\
&= (A \cup B) \cap (\overline{B} \cup \overline{A}) && \text{Identity Law} \\
&= (A \cup B) \cap \overline{(A \cap B)} && \text{DeMorgan's Law} \\
&= (A \cup B) \setminus (A \cap B) && \text{Definition of Set Subtraction}
\end{aligned}
$$

We have thus demonstrated the equality of these sets. $\square$

2.1.1 Suggested Homework

Rosen 2.2: 5–10, 11–17, 18*, 19–20, 32–39, 40*, 41*, 46*, 47–51, 53, 57, 59–60.

3 Friday, February 6

3.1 Set Leftovers

Def: A and B are *disjoint* if $A \cap B = \varnothing$.

Generalized Unions and Intersections:

$$\bigcup_{i=1}^{n} A_i = A_1 \cup A_2 \cup A_3 \cup \cdots \cup A_n.$$

$$\bigcap_{i=1}^{n} A_i = A_1 \cap A_2 \cap A_3 \cap \cdots \cap A_n.$$

Def: A subset R of $A \times B$ is called a *relation* between A and B. The ordered pairs in R describe pairs (a, b) that are related in some way.

Show how to visualize $A \times B$ and a relation using a bipartite graph.

Ex: Let $A = \{1, \ldots, n\}$ and $B = \mathcal{P}(A)$. Let $R = \{(S, T) : S, T \in \mathcal{P}(A), S \subseteq T\}$. Then $R \subseteq \mathcal{P}(A) \times \mathcal{P}(A)$ and the "relation" of a pair $(S, T) \in R$ is "S is a subset of T."

Not all relations are as nicely defined as the one above.

3.2 Rosen 2.3 – Functions

Reading: Rosen 2.3, LLM 4.3, Ducks 3.1, 3.2.

Def: Let A and B be sets. A *function* f from A to B is an assignment of exactly one element of B to each element of A. We write $f(a) = b$ if b is the unique element of B assigned by the function f to the element a of A. If f is a function from A to B, we write $f : A \to B$.

Def: *domain*, *codomain*, *image* ($f(a) = b$), *preimage*, *range*, *maps* A to B.

Draw pictures of the above things.

Examples:

$$f : \mathbb{Z} \to \mathbb{Q}, \quad f(a) = \frac{1}{1 + |a|}.$$

$$f : \mathbb{Z} \to \mathbb{Z}, \quad f(a) = 2a$$
$$f : \mathbb{R} \to \mathbb{R}, \quad f(x) = x^2.$$

$$f : \{a, b, c\} \to \{1, 2, 3, 4\}, \quad f(a) = 1, f(b) = 2, f(c) = 1.$$

Ex: With the finite set examples, discuss how functions create relations, but not all relations create functions.

Def: If $f : A \to B$ and $S \subseteq A$, let $f(S) = \{f(a) : a \in S\}$, and call $f(S)$ the *range of* S.

Def: *integer-valued*, *real-valued*.

Def: Let $f_1, f_2 : \mathbb{R} \to \mathbb{R}$. $(f_1 + f_2)(x) = f_1(x) + f_2(x)$, $(f_1 f_2)(x) = f_1(x) f_2(x)$.

3.2.1 Types of Functions

Def: *injective* (or *one-to-one*), *surjective* (or *onto*), and *bijective*.

Let $f : A \to B$ be a function.

To show that f is injective: Prove $\forall x, y \in A(f(x) = f(y) \to x = y)$.

To show that f is NOT injective: Prove $\exists x, y \in A(f(x) = f(y) \wedge x \neq y)$.

To show that f is surjective: Prove $\forall y \in B \exists x \in A(f(x) = y)$.

To show that f is NOT surjective: Prove $\exists y \in B \forall x \in A(f(x) \neq y)$.

Def: Let f be a bijective function from A to B. The *inverse function* of f is the function, denoted f^{-1}, that assigns to an element $b \in B$ the unique element $a \in A$ such that $f(a) = b$, i.e. $f^{-1}(b) = a$ when $f(a) = b$.

A bijective function is *invertible*, as we can construct the inverse. A function $f : A \to B$ that is not bijective is *not invertible*, since one of two things fails:

- f is not injective, so $f(a) = f(a') = b$ for two elements $a, a' \in A$ where $a \neq a'$. Then $f^{-1}(b)$ cannot be both a and a'.
- f is not surjective, so for some element $b \in B$ there is no element $a \in A$ where $f(a) = b$, so $f^{-1}(b)$ cannot be any element of A.

Def: If $f : A \to B$ and $g : B \to C$ are functions, then the *composition of g and f*, denoted $g \circ f$, is the function $g \circ f : A \to C$ where $(g \circ f)(a) = g(f(a))$ for all $a \in A$.

Draw a picture.

Note: If f is a bijection, then $f^{-1} \circ f : A \to A$ is the *identity map*: $f^{-1} \circ f(a) = a$. Note that $f \circ f^{-1} : B \to B$.

3.2.2 Plots of Functions

A drawing of a real-valued function on the (x, y)-plane is a *plot* of the function. (Not a *graph* because that means something DIFFERENT in this class!)

Ex: $f : \mathbb{Z} \to \mathbb{Z}$: $f(x) = x^2$.

Ex: $f : \mathbb{R} \to \mathbb{R}$: $f(x) = \lfloor x \rfloor$.

Ex: $f : \mathbb{R} \to \mathbb{R}$: $f(x) = \lceil x \rceil$.

Ex: $f : \mathbb{R} \to \mathbb{R}$: $f(x) = [x]$.

Def: The *factorial function* $*! : \mathbb{N} \to \mathbb{Z}^+$, denoted by $n!$, is the function $n(n-1)(n-2)\cdots(3)(2)(1)$ or $\prod_{i=1}^{n} i$. (Note: The empty product is considered to be multiplied by 1, so $0! = 1$. There is a reason for this!)

3.2.3 Suggested Homework

Rosen 2.3: 1–3, 10–15, 20–25, 26*, 28–29, 33–37, 42–44, 47, 77.