1 Monday, March 9

1.1 Rosen 6.2: Pigeonhole Principle

Reading: Rosen 6.2, Ducks 1.5, LLM

> **Pigeonhole Principle.**
> If k is a positive integer and $k+1$ or more objects are placed into k boxes, then there is at least one box containing two or more objects.

Note: The pigeonhole principle does *not* say that if we place $k+1$ or more objects into k boxes that every box has at least one object! We could put all objects into one box!

Ex: Let f be a function from a set A to a set B. If $|A| > |B|$, then f is not injective.

Ex: Among a group of 367 people, there must be two with the same birthday, as there are only 366 possible birthdays.

Ex: In any group of 27 English words, there are two that start with the same letter and there are two that end with the same letter.

Ex: In any group of 677 English words, there are two that start with the same letter and end with the same letter.

Ex: Among any $n+1$ positive integers not exceeding $2n$ there must be an integer that divides one of the other integers.

Proof. Let $\{a_1, \ldots, a_n, a_{n+1}\}$ be integers such that $1 \le a_i \le 2n$ for all $i \in \{1, \ldots, n\}$. There exist natural numbers k_i and odd integers q_i such that $a_i = 2^{k_i} q_i$ for each i. Note that $1 \le q_i \le 2n-1$ for each odd integer q_i, so there are only n possible odd integers in that range. Thus there are two numbers a_i and a_j where $q_i = q_j$. Then, either $k_i \le k_j$, or $k_i > k_j$ (w.l.o.g. assume $k_i \le k_j$). Then $a_j = 2^{k_j} q_j = 2^{k_j - k_i}(2^{k_i} q_i) = 2^{k_j - k_i} a_i$ and hence a_i divides a_j. □

Ex: A sociologist studies random groups of n people on Facebook, and studies how many friends each person has within the group of n people. She notices a pattern: there is always a pair of people who have the same number of friends in the group. Why?

Proof. Let $p_1, \ldots, p_n$ be n people on Facebook. For $i \in \{1, \ldots, n\}$, let $d(p_i)$ be the number of people within the set $\{p_1, \ldots, p_n\}$ that are friends with p_i. Note that $0 \le d(p_i) \le n-1$, so there are a total of n options. *However,* if $d(p_i) = n$ for some i, then p_i is friends with everyone in the group! If $d(p_j) = 0$ for some i, then p_j is not friends with anyone in the group! Thus, we cannot simultaneously have $d(p_i) = n$ and $d(p_j) = 0$ for some i and j. So there really are $n-1$ values that each $d(p_i)$ can take, but that means we have n inputs to the function d and $n-1$ outputs, so two of the inputs, say p_i and p_j, must have the same output $d(p_i) = d(p_j)$. □

> **Generalized Pigeonhole Principle.**
> If N objects are placed into k boxes, then there is a box containing at least $\lceil N/k \rceil$ objects.

Proof. Let N and k be positive integers. Let $N = qk - r$ where $0 \le r < k$, so $\lceil N/k \rceil = q = (N+r)/k$. Suppose that we place objects into k boxes and every box contains at most $\lceil N/k \rceil - 1$ objects. Then, the number of objects we placed was at most

$$k(\lceil N/k\rceil - 1) = k\lceil N/k\rceil - k = k((N+r)/k) - k = (N+r) - k < N + k - k = N.$$

Therefore, we placed fewer than N objects. $\square$

Ex: If there are 80 students in this class, how many are guaranteed to have their birthday in the same calendar month? (A: $\lceil 80/12\rceil = 7$)

Def: Let n be an integer with $n \geq 2$. Let E_n be the set of pairs $\{i, j\}$ where $1 \leq i < j \leq n$. A *2-coloring* is an assignment $c : E_n \to \{\text{red}, \text{blue}\}$. For integers a, b *Ramsey number* $r(a, b)$ is the minimum n such that for every 2-coloring $c : E_n \to \{\text{red}, \text{blue}\}$ there either exists a set $\{x_1, \ldots, x_a\} \subset \{1, \ldots, n\}$ where every pair $\{x_i, x_j\}$ is colored red, or there exists a set $\{y_1, \ldots, y_b\} \subset \{1, \ldots, n\}$ where every pair $\{y_i, y_j\}$ is colored blue.

Thm: $r(3, 3) = 6$.

Proof. To prove $r(3, 3) > 5$, we demonstrate a coloring. (Shown in class, but not in notes.)

To prove $r(3, 3) \leq 6$, we will prove that every 2-coloring of E_6 contains a red "triangle" or a blue "triangle". Fix a 2-coloring $c : E_6 \to \{\text{red}, \text{blue}\}$. Consider the element 1. Among the five pairs $\{1, i\}$ for $2 \leq i \leq 6$, there are three of the same color, by generalized pigeonhole principle. Let $2 \leq i < j < k \leq 6$ be the value such that $\{1, i\}$, $\{1, j\}$, $\{1, k\}$ have the same color, (w.l.o.g. the color is red). Among the three pairs $\{i, j\}$, $\{i, k\}$, and $\{j, k\}$, if there is a red pair, then we have a red triangle! Therefore, they must all be blue, but that forms a blue triangle! $\square$

For more information, see `http://en.wikipedia.org/wiki/Ramsey%27s_theorem`

Def: Given a list $a_1, \ldots, a_n$ of n distinct real numbers, an *increasing subsequence* is a list $a_{i_1}, \ldots, a_{i_k}$ where $i_1 < i_2 < \cdots < i_k$ and $a_{i_1} < a_{i_2} < \cdots < a_{i_k}$; a *decreasing subsequence* is a list $a_{i_1}, \ldots, a_{i_k}$ where $i_1 < i_2 < \cdots < i_k$ and $a_{i_1} > a_{i_2} > \cdots > a_{i_k}$. In these subsequences, k is the *length.*

Erdős-Szekeres Theorem: Every sequence of n^2+1 distinct real numbers contains a subsequence of length $n + 1$ that is either strictly increasing or strictly decreasing.

Proof. Let $a_1, \ldots, a_{n^2+1}$ be a list of $n^2 + 1$ distinct real numbers and suppose for the sake of contradiction that the list contains no increasing subsequence of length $n+1$ or decreasing subsequence of length $n+1$. For each $k \in \{1, \ldots, n^2+1\}$, let (i_k, d_k) be the pair where i_k is the maximum length of an increasing subsequence ending at a_k and d_k is the maximum length of a decreasing subsequence ending at a_k. Since each number by itself is an increasing or decreasing subsequence, $i_k \geq 1$ and $d_k \geq 1$. Since no increasing or decreasing subsequence of length $n + 1$ exists, $i_k \leq n$ and $d_k \leq n$. Thus, each of these pairs (i_k, d_k) is in the set $\{1, \ldots, n\} \times \{1, \ldots, n\}$ of size n^2. By the pigeonhole principle, there exist two distinct values $k < \ell$ where $(i_k, d_k) = (i_\ell, d_\ell)$. However, $a_k \neq a_\ell$, so either $a_k < a_\ell$ or $a_k > a_\ell$.

If $a_k < a_\ell$, then there is an increasing subsequence of length i_k ending at a_k, and that subsequence has all previous terms smaller than a_k. These terms are all smaller than a_ℓ, so there is an increasing subsequence of length $i_k + 1$ ending at a_ℓ, contradicting the definition of $i_\ell = i_k$.

If $a_k > a_\ell$, then there is a decreasing subsequence of length d_k ending at a_k, and that subsequence has all previous terms larger than a_k. These terms are all larger than a_ℓ, so there is a decreasing subsequence of length $d_k + 1$ ending at a_ℓ, contradicting the definition of $d_\ell = d_k$.

Thus, we have found a contradiction in all cases, so the assumption that $a_1, \ldots, a_{n^2+1}$ contains no increasing or decreasing subsequence of length $n + 1$ is false. $\square$

2 Wednesday, March 11

2.1 Rosen 6.4: Binomial Coefficients and Identities

Reading: Rosen 6.4, Ducks , LLM

Recall our definition of the *binomial coefficient* $\binom{n}{k}$, stated as "n choose k," to be the number of k-combinations among a set of n distinct objects.

Binomial Theorem: Consider the polynomial $p_n(x,y) = (x+y)^n$. The coefficient of $x^k y^{n-k}$ in $p(x,y)$ is $\binom{n}{k}$. That is, $(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}$.

Proof. When distributing the product $\underbrace{(x+y)(x+y)\cdots(x+y)}_{n \text{ terms}}$, we can produce the $x^k y^{n-k}$ term by multiplying k of the "x side" and $n-k$ of the "y side." There are $\binom{n}{k}$ ways to select which terms we will use the "x side" and each $x^k y^{n-k}$ term is added resulting in a coefficient of $\binom{n}{k}$. □

Don't like the proof above? Try another one!

Proof. We prove by induction on $n \geq 0$ that for all k, the kth term of $p_n(x,y) = (x+y)^n$ is $\binom{n}{k} x^k y^{n-k}$.

Case $n = 0$: $p_0(x,y) = 1$, so the coefficient of $x^0 y^0$ is 1.

(Induction Hypothesis) Let $n \geq 0$ and suppose that $(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}$.

Case $n+1$: Consider how the term $x^k y^{n+1-k}$ can appear in $p_{n+1}(x,y)$. But first realize that $p_{n+1}(x,y) = (x+y)^{n+1} = (x+y)p_n(x) = xp_n(x,y) + yp_n(x,y)$. Therefore, $x^k n^{n+1-k}$ can appear as a term on the left or right of the sum. On the right, the coefficient of $x^k y^{n-k}$ in $p_n(x,y)$ is $\binom{n}{k}$ by the Induction Hypothesis, and thus the coefficient of $x^k y^{(n+1)-k} = y \cdot x^k y^{n-k}$ of $yp_n(x,y)$ is $\binom{n}{k}$. On the left, the coefficient of $x^{k-1} y^{n-(k-1)}$ in $p_n(x,y)$ is $\binom{n}{k-1}$ by the Induction Hypothesis, and thus the coefficient of $x^k y^{n-(k-1)} = x \cdot x^{k-1} y^{(n+1)-k}$ of $xp_n(x,y)$ is $\binom{n}{k-1}$. Therefore, the coefficient of $x^k y^{(n+1)-k}$ in $p_{n+1}(x)$ is $\binom{n}{k} + \binom{n}{k-1}$. By Pascal's Identity, $\binom{n}{k} + \binom{n}{k-1} = \binom{n+1}{k}$. □

Thm: $\sum_{k=0}^{n} \binom{n}{k} = 2^n$.

Proof 1. By the Binomial Theorem, $2^n = (1+1)^n = \sum_{k=0}^{n} \binom{n}{k} 1^k 1^{n-k} = \sum_{k=0}^{n} \binom{n}{k}$. □

Proof 2. Let A be a set of size n. There are 2^n subsets of A (by product rule, selecting a subset by deciding containment of each element of A in order). There are $\binom{n}{k}$ k-subsets of A, so there are $\sum_{k=0}^{n} \binom{n}{k}$ subsets of A. □

Thm: $\sum_{k=0}^{n} (-1)^k \binom{n}{k} = 0$.

Proof 1. By the Binomial Theorem, $0^n = ((-1)+1)^n = \sum_{k=0}^{n} \binom{n}{k} (-1)^k 1^{n-k} = \sum_{k=0}^{n} (-1)^k \binom{n}{k}$. □

Proof 2. (Note: This proof holds only when n is odd, so k and $n-k$ have opposite parities. The last equality is subtle.)

Since $\binom{n}{k} = \binom{n}{n-k}$, we have $\binom{n}{k} - \binom{n}{n-k} = 0$. Therefore, $0 = \sum_{k=0}^{\lfloor n/2 \rfloor} (-1)^k (\binom{n}{k} - \binom{n}{n-k}) = \sum_{k=0}^{n} (-1)^k \binom{n}{k}$. □

Thm: $\sum_{k=0}^{n} 2^k \binom{n}{k} = 3^n$.

Proof 1. By the Binomial Theorem, $3^n = (2+1)^n = \sum_{k=0}^{n} \binom{n}{k} 2^k 1^{n-k} = \sum_{k=0}^{n} 2^k \binom{n}{k}$. $\square$

Proof 2. Let A be a set of size n. We will count the number of sets B, C where $C \subseteq B \subseteq A$.

For each element a of A, we can decide three options: a is not in B or C, a is in B, or a is in B and C. (Note, we cannot have a in C and a not in B, or else $C \nsubseteq B$.) Thus, there are 3^n situations.

For each subset B of size k, there are 2^k subsets $C \subseteq B$. Thus, there are $\sum_{k=0}^{n} 2^k \binom{n}{k}$ pairs $C \subseteq B$. $\square$

Vandermonde's Identity: Let m, n, and r be nonnegative integers with $r \leq \min\{n, m\}$.

$$\binom{m+n}{r} = \sum_{k=0}^{n} \binom{m}{r-k} \binom{n}{k}.$$

Proof. Let A be a set of n Cyclone fanes and B a set of m Hawkeye fans. **Obviously** no one is both a Cyclone fan and a Hawkeye fan.

There are $\binom{m+n}{r}$ ways to invite r people from $A \cup B$ to a tailgating party.

In designing my tailgating party, I could first specify that I want k Cyclone fans and $r-k$ Hawkeye fans. There are $\binom{n}{k}\binom{m}{r-k}$ ways to invite k Cyclone fans and $r-k$ Hawkeye fans to my tailgating party. By the sum rule, there are $\sum_{k=0}^{r} \binom{m}{r-k}\binom{n}{k}$ ways to invite people from $A \cup B$ to my party. $\square$

Cor: $\binom{2n}{n} = \sum_{k=0}^{n} \binom{n}{k}^2$.

Proof. By Vandermonde's identity with $r = m = n$, we have $\binom{2n}{n} = \binom{n+n}{n} = \sum_{k=0}^{n} \binom{n}{n-k}\binom{n}{k}$ but by Pascal's identity ($\binom{n}{n-k} = \binom{n}{k}$) we complete the proof. $\square$

3 Friday, March 13

EXAM 2.