MATH 484 - Final Exam

Section D13 D14 Name: ....

Date: Dec 13 2011

Work on your own. Write clearly. Ask if something is not clear. If you need more paper, let me know. Good luck!

This is not exam - it just contains some list of definitions and theorems. It will be updated over time. If you see some a mistake or a typo, please let me know.
This version is from: 12:12 December 12, 2011. It should contain all theoretical questions. Of course, there will be many computational questions on the exam too!

Question 1:

Write definitions of the following terms:
- (global,local)(strict)minimizer and maximizer of a real function *page 2*
- critical point of a real function *page 2*
- cosine of two vectors *page 6*
- ball $B(\mathbf{x}, r)$ (what is $\mathbf{x}$ and r?) *page 6*
- interior D^0 of set $D \subseteq R^n$ *page 6, page 164*
- open set $D \subseteq R^n$ *page 6*
- closed set $D \subseteq R^n$ *page 7*
- compact set $D \subseteq R^n$ *page 6*
- (global,local)(strict)minimizer and maximizer of a function $f : \mathbb{R}^n \to \mathbb{R}$ *page 8*
- critical point of a function $f : \mathbb{R}^n \to \mathbb{R}$ *page 8*
- gradient $\nabla f(\mathbf{x})$ where $f : \mathbb{R}^n \to \mathbb{R}$ *page 10*
- Hessian $Hf(\mathbf{x})$ where $f : \mathbb{R}^n \to \mathbb{R}$ *page 10*
- quadratic form associated with a matrix A *page 12*
- (positive,negative)(semi)definite matrix *page 13*
- indefinite matrix *page 13*
- Δ_k, the kth principal minor of a matrix A *page 16*
- $f : \mathbb{R}^n \to \mathbb{R}$ being coercive *page 25*
- eigenvalues and eigenvectors of a matrix A *page 29*
- $C \subset R^n$ being convex *page 38*
- closed and open half-spaces in R^n *page 40*
- convex combination of k vectors from R^n *41*
- convex hull of $D \subseteq R^n$ *42*
- (strictly) convex and concave function $f : C \to \mathbb{R}$, where $C \subseteq \mathbb{R}^n$ *page 49*
- posynomial *page 67*
- primal and dual geometric program *page 67,68*
- best least squares kth degree polynomial *page 135*
- linear regression line *page 135*
- best least squares solution of (inconsistent) linear system *page 136*
- generalized inverse of a matrix A *page 136*
- inconsistent linear system *page 136–137*
- orthonormal vectors *page 138*
- subspace of $\mathbb{R}^n$ *page 141*
- orthogonal complement of a subspace of $\mathbb{R}^n$ *page 142*
- P_M - orthogonal projection of R^m onto M *page 144*
- underdetermined system of linear equations *page 145*
- H-inner product *page 149*
- H-norm *page 149*

- H-orthogonal vectors *page 149*
- H-orthogonal complement *page 149*
- H-generalized inverse*page 150*
- hyperplane H in $\mathbb{R}^n$ *page 158*
- boundary point of $C \subset \mathbb{R}^n$ *page 158*
- closure $\overline{A}$ of $A \subset \mathbb{R}^n$ *page 163*
- subgradient of $f : \mathbb{R}^n \to \mathbb{R}$ *page 168*
- subdifferential of $f : \mathbb{R}^n \to \mathbb{R}$ *page 168*
- feasible vector of a program (P) *page 169*
- feasible region of a program (P) *page 169*
- consistent program (P) *page 169*
- superconsistent program (P) *page 169*
- convex program and dual convex program *pages 169, 200, 201*
- supremum of a real valued function defined on $C \subseteq R^n$ *page 170*
- infimum of a real valued function defined on $C \subseteq R^n$ *page 170*
- MP for program (P) - also define (P) *page 171*
- $MP(z)$ for program $(P(z))$ - also define $(P(z))$ *page 171*
- sensitivity vector of a program (P) *page 177*
- Lagrangian $L(\mathbf{x}, \lambda)$ of a program (P) - also define (P) *page 182*
- complementary slackness conditions - also define (P) *page 184*
- constrained geometric program (GP) and its dual (DGP) *page 193*
- linear program (LP) and its dual (DLP) *pages 173,201,202*
- duality gap *page 209*
- Absolute value penalty function *page 217*
- penalty parameter *page 217*
- Courant-Beltrami penalty function *page 219*
- generalized penalty function *page 223*
- exact penalty function *page 226*
- program (P^ϵ) *page 230*
- $Tr(A)$ - a trace of a matrix A *[SDP notes]*
- primal and dual form of a semidefinite program (SDP) *[SDP notes]*

Question 2:

State theorems which give answers to the following questions: *(without proofs)*

- What are the implications of first and second derivatives on minimizers and maximizers of $f : \mathbb{R} \to \mathbb{R}$? *Theorem 1.1.5*
- What are the implications of first and second partial derivatives on minimizers and maximizers of a function $f : \mathbb{R}^n \to \mathbb{R}$? *Theorem 1.2.5*
- What are implications of definiteness $Hf(x)$ on global minimizers and maximizers? *Therorem 1.2.9*
- What are implications of definiteness $Hf(x)$ on local minimizers and maximizers? *Therorem 1.3.6*
- What can you say about extremes of a coercive function? *Theorem 1.4.4*
- What is the relationship between eigenvalues and positive(negative)(semi)definiteness of a symmetric matrix A? *Theorem 1.5.1*
- What is the relation between convex hull and set of all convex combinations of vectors from $D \subseteq R^n$? *Theorem 2.1.4*
- What do you know about minimizers of (strictly) convex functions (in $\mathbb{R}^n$)? *Theorem 2.3.4, Corollary 2.3.6*
- Is there relationship between begin s (strictly) convex function and having continuous first partial derivatives (in $\mathbb{R}^n$)? *Theorem 2.3.5*
- Is there relationship between begin (strictly) convex function and having continuous second partial derivatives (condition using $Hf(\mathbf{x})$) (in $\mathbb{R}^n$)? *Theorem 2.3.7*
- State Aithmetic-Geometric Mean Inequality. Include also when it is equality! *Theorem 2.4.1*
- State duality theorem for geometric programs. *Theorem 2.5.2*
- What does and how to compute P_M? (orthogonal projection of R^m onto M) *Theorem 4.2.5*
- What is the form of solutions of underdetermined systems? *Theorem 4.3.1*
- What is the form of minimum norm solutions of underdetermined systems? *Theorem 4.3.2*
- What is the form of minimum H-norm solutions of underdetermined systems? *Theorem 4.4.2*
- What is the way of computing of the closest vector of a convex set to a given vector? *Theorem 5.1.1*
- What is the characterization of the closest vector of a convex set to a given vector using orthogonal complement? *Theorem 5.1.2*
- What is a sufficient condition for existence of a closest vector from a set C to a given vector $\mathbf{x}$? *Theorem 5.1.3*
- What is a sufficient condition for existence of a unique closest vector from a set C to a given vector $\mathbf{x}$? *Corollary 5.1.4*
- State basic separation theorem. *Theorem 5.1.5*
- State Support theorem. *Theorem 5.1.9*
- What can you say about $MP(z)$ if (P) is super consistent? *(Theorem 5.2.6)*
- Can MP be computed from the sensitivity vector? *(Theorem 5.2.11)*
- State Karush-Kuhn-Tucker Theorem (Saddle point version) *Theorem 5.2.13*
- State Karush-Kuhn-Tucker Theorem (Gradient form) *Theorem 5.2.14*
- State Extended Arithmetic-Geometric Mean Inequality Include also when it is equality! *Theorem 5.3.1*
- What are sufficient condition for a constrained geometric program (GP) to have no duality gap? *Theorem 5.3.5*
- State the duality theorem of linear programming *page 203*
- State duality theorem of convex programming *Theorem 5.4.6*
- What are sufficient conditions for a constrained convex program (P) to have no duality gap? *Theorem 6.3.1*
- State the duality theorem for semidefinite programming *SDP notes - first theorem*
- What do you know about solvability of SDP? *SDP notes - second theorem*

Question 3:

Answer the following:

- Is product of two convex functions convex?
- How to express best least square solution using QR factorization? What is advantage of QR over using generalized inverse?
- Describe the intuition using angles behind the Theorem stating what is the closest vector from a convex set to a given vector. *Theorem 5.1.1, page 159*
- Does every convex set always contain a vector that is closest to a given vector? Why?
- Is $MP(z)$ always continuous?
- Why do we consider MP and MD instead of min and max?
- What is relation between KKT multipliers and sensitivity vector?
- Why is it necessary to use extended (AG) for bounding $g_i(t)$ while solving constrained geometric programs?
- How can you (in theory) try to solve geometric program using its dual? *(page 201)* - What is a disadvantage of Courant-Beltrami penalty function? *Answer: "isn't exact" - but more details are expected ;-)*
- What are advantages (or consequences) of having no duality gap? *Answer: "primal-dual algorithm, algorithms with guaranteed performance, certificate of optimality" - but more details are expected - like what is what ;-)*
- Describe how is it possible to change the objective function of a convex program such that the objective is coercive. What is the reason for doing it? *pages 229,230*
- Can be ANY linear program expressed as SDP?
- Why is semidefinite programming important? *Answer hints: What can you express as SDP? Can you solve it?* - Give example how can you use the condition that a matrix is positive semidefinite while trying to express a problem as semidefinite program. *Answer: You can use "quadratic constraint" and write express it as $x * y - z^2$ and this "corresponds" to determinant of a 2×2 matrix. Or if you can express your variables as vectors, you may use that every positive semidefinite matrix A has a unique decomposition $A = U^T U$. See SDP notes for both of these two. Maybe you can give a small example.*

Question 4: [Computational question on SDP will look like this:]
Express the following program as a semidefinite program. *(Do not solve the resulting program.) See SDP notes for examples and your notes for* $(LP) \to (SDP)$ *example.*

Question X: D14 only *(D13 may try too if they wish)*

- State and prove the theorem relating local and global minimizers of a convex function (in $\mathbb{R}^n$). *Theorem 2.3.4*
- State and prove Arithmetic-Geometric Mean Inequality. Include also when it is equality. *Theorem 2.4.1*
- State and prove basic separation theorem. *Theorem 5.1.5*
- State and prove Support theorem. *Theorem 5.1.9*
- Derive dual geometric program from primal using AG inequality. *page 67–68*
- State and prove Karush-Kuhn-Tucker Theorem in Saddle point version. *Theorem 5.2.13*
- State and prove extended arithmetic geometric mean inequality. *Theorem 5.3.1*