Math-484 Homework #7

I will finish this homework before 11 am Oct 19 and bring it to class. If I have troubles with my work I may come to the study session on Oct 17, 5-7 pm, 145 Altgeld Hall. If I spot a mathematical mistake I will let the lecturer know as soon as possible.
I will write clearly and neatly as the grader is not an expert in cryptography. I will sign each paper of my work and indicate if I am D14 (4 hours student).

Exercise 1: *(Can I work with H-norm?)*
Minimize $f(x, y, z) = 2x^2 + 2xy + 2y^2 + z^2$
subject to $x - y + z = 3$
$9x + 6y + 2z = 5$

Exercise 2: *(I want to know what if* epigraph.*)*
Let $D \subset \mathbb{R}^n$ be convex and $f : D \to \mathbb{R}$. The *epigraph* of f is a subset of $\mathbb{R}^{n+1}$ defined by

$$epi(f) = \{(\mathbf{x}, \alpha) : \mathbf{x} \in D, \alpha \in \mathbb{R}, f(\mathbf{x}) \leq \alpha\}.$$

a) Sketch the epigraphs of functions
$f(x) = e^x$ for $x \in \mathbb{R}$
$f(x_1, x_2) = x_1^2 + x_2^2$ for $(x_1, x_2) \in \mathbb{R}^2$
b) Show that $f(\mathbf{x})$ is convex if and only if epi(f) is convex.
c) Show that if $f(\mathbf{x})$ and $g(\mathbf{x})$ are convex functions defined on a convex set C then $h(\mathbf{x}) := \max\{f(\mathbf{x}), g(\mathbf{x})\}$ is also a convex function on C by showing that

$$\text{epi}(\max\{f(\mathbf{x}), g(\mathbf{x})\}) = \text{epi}(f(\mathbf{x})) \cap \text{epi}(g(\mathbf{x})).$$

Exercise 3: *(More about orthogonal complements)*
Let M be a subspace of $\mathbb{R}^n$. Prove that the orthogonal complement $M^\perp$ of M is closed.

Exercise 4: *(What is an interior?)*
Prove that if M is a subspace of $\mathbb{R}^n$ such that $M \neq \mathbb{R}^n$, then the interior M^0 of M is empty.

Exercise 5: *(What are closest vectors?)*
Let C be a closed convex subset of $\mathbb{R}^n$. If $\mathbf{y} \notin C$, show that $\mathbf{x}^* \in C$ is the closest vector to $\mathbf{y}$ in C if and only if $(\mathbf{x} - \mathbf{y}) \cdot (\mathbf{x}^* - y) \geq ||\mathbf{x}^* - \mathbf{y}||^2$ for all $\mathbf{x} \in C$.

Exercise 6: *(Do I remember ancient stuff?)*
Show that

$$\left(\frac{x}{2} + \frac{y}{3} + \frac{z}{12} + \frac{w}{12}\right)^4 \leq \frac{1}{2}x^4 + \frac{1}{3}y^4 + \frac{1}{12}z^4 + \frac{1}{12}w^4$$

holds with equality if and only if $x = y = z = w$.

Exercise 7: *(Can I separate things? **D14 only**)*
Let $C_1, C_2 \subset \mathbb{R}^n$ be both convex, $C_1^0 \neq \emptyset$ and $C_1^0 \cap C_2 = \emptyset$. Prove that there exist $0 \neq \mathbf{a} \in \mathbb{R}^n$ and $\alpha \in \mathbb{R}$ such that

$$\mathbf{x}^T\mathbf{a} \leq \alpha \leq \mathbf{y}^T\mathbf{a}$$

for all $\mathbf{x} \in C_1$ and $\mathbf{y} \in C_2$. In other words, there is a hyperplane separating C_1 and C_2 (but both C_1 and C_2 can intersect the hyperplane).
