SEMIDEFINITE PROGRAMMING

DEFINITION

LET $A \in \mathbb{R}^{m \times m}$. THE TRACE OF A IS

$$tr(A) = \sum_{i=1}^{m} a_{ii}$$

$$SYM_m = \{ X \in \mathbb{R}^{m \times m} : X_{ij} = X_{ji} \quad \forall i,j \}$$

SYMMETRIC MATRICES $m \times m$

RECALL (LP)

$$(LP) \begin{cases} \max & c^T x \\ \text{s.t.} & Ax = b \\ & x \geq 0 \end{cases}$$

$x = (x_1 \ldots x_n)$, $c \in \mathbb{R}^n$, $b \in \mathbb{R}^m$, $A \in \mathbb{R}^{m \times n}$

$$(LP)\begin{cases} \max & C \cdot X \\ \text{S.T.} & a_1 \cdot X = b_1 \\ & a_2 \cdot X = b_2 \\ & \vdots \\ & a_m \cdot X = b_m \\ & X \geq 0 \end{cases}$$

a_i .. i-TH ROW OF A

$c, x, a_1 \ldots a_m$ ARE VECTORS $\in \mathbb{R}^n$

$b_1, b_2, \ldots b_m \in \mathbb{R}$

REPLACE • VECTOR SPACE $\mathbb{R}^m$ BY VECTOR SPACE

SYM_m

- THE DOT PRODUCT BY

$$X \cdot Y = \langle X | Y \rangle = T_r(X^T Y) =$$
$$= \sum_{i=1}^{m} \sum_{j=1}^{m} X_{ij} Y_{ij}$$

- $x \geq 0$ BY X IS POSITIVE SEMIDEFINITE. DENOTED BY

$$X \succeq 0$$

($X \succ 0$, X IS POSITIVE DEFINITE)

$$
(SDP)\begin{cases}
\text{MAX} & Tr(C^T X) \\
\text{S.T.} & Tr(A_1^T X) = b_1 \\
& \vdots \\
& Tr(A_m^T X) = b_m \\
& X \succeq 0
\end{cases}
$$

$C, A_1 \ldots A_m, X \in SYM_n$

$b_1 \ldots b_m \in \mathbb{R}$

$$
Tr\left(\begin{pmatrix} C_{11} & C_{12} \\ C_{12} & C_{22} \end{pmatrix}^T \begin{pmatrix} X_{11} & X_{12} \\ X_{12} & X_{22} \end{pmatrix}\right) = C_{11}X_{11} + 2C_{12}X_{12} + C_{22}X_{22}
$$

→ EQUIVALENT FORM OF (SDP)

$$
(SDP)\begin{cases}
\text{MAX} & \sum_{i \leq j} c_{ij} x_{ij} \\
\text{S.T.} & \sum_{i \leq j} a_{ijk} x_{ij} = b_k, \quad k = 1 \ldots m \\
& X \succeq 0
\end{cases}
$$

(LP) IS SPECIAL CASE OF (SDP)

$$c \longrightarrow \begin{pmatrix} c_1 & & 0 \\ & c_2 & \\ 0 & & \ddots \\ & & & c_m \end{pmatrix} = C$$

$$a_i \longrightarrow \begin{pmatrix} a_{i1} & & 0 \\ & a_{i2} & \\ 0 & & \ddots \\ & & & a_{im} \end{pmatrix} = A_i$$

$$x \longrightarrow \begin{pmatrix} x_1 & & 0 \\ & x_2 & \\ 0 & & \ddots \\ & & & x_n \end{pmatrix} = X$$

NOTE: $x \geq 0 \iff X \succeq 0$

(x_i ARE EIGENVALUES OF X)

NOTE HOW TO HANDLE INEQUALITY?

$$A_i \cdot X \geq b_i$$

EXTEND X:

$$\begin{pmatrix} A_i & 0 \\ 0 & -1 \end{pmatrix} \cdot \begin{pmatrix} X & 0 \\ 0 & x_i' \end{pmatrix} = b_i$$

AS x_i' IS ANYTHING ≥ 0

DUAL FORM:

$$(\text{DSDP}) \begin{cases} \text{MINIMIZE} \quad b^T \cdot y \\ \text{ST.} \\ \quad y_1 \cdot A_1 + y_2 A_2 + \dots y_n A_n - C \succeq 0 \end{cases}$$

DEF:

- (SDP) IS STRICTLY FEASIBLE IF $\exists$ FEASIBLE X WHICH IS POSITIVE DEFINITE $(X \succ 0)$
- (DSDP) IS STRICTLY FEASIBLE IF $\exists$ FEASIBLE y ST. $(y \cdot A - C \succ 0)$

THEOREM (STRONG DUALITY OF SDP)

- IF (SDP) IS STRICTLY FEASIBLE AND HAS OPT. SOLUTION OF VALUE γ THEN (DSDP) IS FEASIBLE AND HAS OPT. SOLUTION OF VALUE γ
- IF (DSDP) IS STRICTLY FEASIBLE AND HAS OPT. SOLUTION OF VALUE γ THEN (SDP) IS FEASIBLE AND HAS OPT. SOLUTION OF VALUE γ

THEOREM (SOLVABLE IN PTIME)

LET (SDP) BE FEASIBLE, FEASIBLE REGION F OF (SDP) BE BOUNDED, LET $R \in \mathbb{N}$ BE SUCH THAT $R \geq \sqrt{Tr(X^T X)}$ $\forall X \in F$ ($\|X\|_2 = \sqrt{X \cdot X}$) AND $\varepsilon > 0$. LET m BE SIZE OF (SDP) [BINARY ENCODED INPUT] THEN IN PTIME WE CAN COMPUTE X' OF VALUE OPTIMUM $- \varepsilon$.

EXAMPLE .. (SDP) WHERE $\boxed{\text{ALL}}$ FEASIBLE VECTORS ARE $\boxed{\text{HUGE}}$

(DSDP);

$$X = \begin{pmatrix} 1 & 2 & & & & & & & \\ 2 & x_1 & & & & & & & \\ & & 1 & x_1 & & & & & \\ & & x_1 & x_2 & & & & & \\ & & & & 1 & x_2 & & & \\ & & & & x_2 & x_3 & & & \\ & & & & & & \ddots & & \\ & & & & & & & 1 & x_{n-1} \\ & & & & & & & x_{n-1} & x_n \end{pmatrix} \succeq 0$$

$$x_1 \begin{pmatrix} 0 & 0 & & & & \\ 0 & 1 & 0 & & & 0 \\ & 0 & 0 & 1 & & \\ & & 1 & 0 & & \\ & & & & & \\ & 0 & & & & \end{pmatrix} + x_2 \ldots + \begin{pmatrix} 1 & 2 & & & \\ 2 & 0 & & & \\ & & 1 & & \\ & & & 0 & \\ & & & & 1 \\ & & & & & \ddots \end{pmatrix}$$

IF $X \succeq$ THEN

$$\begin{pmatrix} 1 & x_i \\ x_i & x_{i+1} \end{pmatrix} \succeq 0 \implies$$

$$\Rightarrow \begin{vmatrix} 1 & x_{i-1} \\ x_{i-1} & x_i \end{vmatrix} \geq 0 \Rightarrow$$

$$x_i - x_{i-1}^2 \geq 0$$

$$x_1 \geq 4 = 2^2$$

$$x_2 \geq x_1^2 = \left(\left(2^2\right)^2\right)^{m} = 2^{2^m}$$

$\Rightarrow \log_2 2^{2^m} = 2^m \rightarrow$ JUST WRITING SOLUTION REQUIRES EXP. TIME!

EXAMPLE

$$\text{MINIMIZE } \frac{(c^T x)^2}{d^T x}$$

$$\text{S.T. } Ax + b \geq 0$$

WHERE $d^T x > 0$ WHENEVER $Ax + b \geq 0$

$x, c, d \in \mathbb{R}^n \quad A \in \mathbb{R}^{m \times n} \quad b \in \mathbb{R}^m$

WRITE AS (DSDP)

1) LINEAR OBJECTIVE FUNCTION

MINIMIZE t

S.T. $Ax + b \geq 0$

$$\frac{(c^T x)^2}{d^T x} \leq t$$

$$(c^T x)^2 \leq d^T x \cdot t$$

$$d^T x \cdot t - (c^T x)^2 \geq 0$$

$$\begin{vmatrix} t & c^T x \\ c^T x & d^T x \end{vmatrix} \geq 0$$

MIN t

$$\begin{pmatrix} a_1 \cdot x + b_1 & & & & \\ & a_2 \cdot x + b_2 & & 0 & \\ & & \ddots & & \\ & & & a_m x + b_m & \\ & 0 & & & \begin{matrix} t & c^T x \\ c^T x & d^T x \end{matrix} \end{pmatrix} \geq 0$$

$\boxed{\text{MAXCUT}}$

DEF: GRAPH

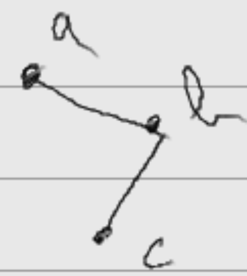
$G = (V, E)$

V ... VERTICES

$E \subseteq \binom{V}{2}$ .. EDGES

$V = \{a, b, c\}$

$E = \{\{a,b\}, \{b,c\}\}$



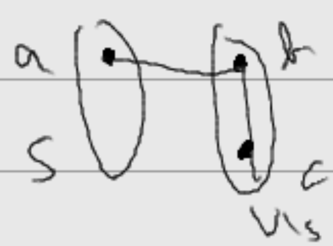
MAXCUT PROBLEM

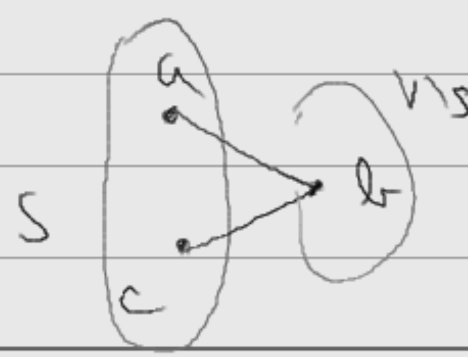
INPUT: G

OUTPUT $S \subseteq V$ S.T.

$$\left|\{e \in E : |e \cap S| = |e \cap (V \setminus S)| = 1\}\right| \text{ IS}$$

MAXIMIZED





EXAMPLE: QUADRATIC CONSTRAINT

$$f(x) = (Ax+b)^T (Ax+b) - c^T x - d \leq 0$$

$$A \equiv [a_1 \; a_2 \; \dots \; a_k]$$

$$c, x \in \mathbb{R}^k, \quad b, d \in \mathbb{R},$$

DUAL FORM

$$F(x) = F_0 + x_1 F_1 + \dots + x_k F_k$$

$$F_0 = \begin{pmatrix} 1 & b \\ b & d \end{pmatrix}, \quad F_i = \begin{pmatrix} 0 & a_i \\ a_i & c_i \end{pmatrix}$$

$$f(x) = \begin{pmatrix} 1 & b + \sum x_i a_i \\ b + \sum x_i a_i & d + \sum c_i x_i \end{pmatrix} \geq 0$$

$$(d + \sum c_i x_i) - (b + \sum x_i a_i)^2 \geq 0$$

det(F(x))

$$\begin{aligned} \text{MIN} \quad & f_0(x) \\ \text{ST} \quad & f_1(x) \le 0 \\ & f_2(x) \le 0 \\ & f_L(x) \le 0 \end{aligned}$$

$$f_i(x) = (A_i x + b_i)^2 - c_i^T x - d_i$$

$$f_0(x) \le t \quad \& \quad \text{MIN } t$$

$$(A_0 x + b_0)^2 - c_0^T x - d_0 - t \le 0$$

$$\text{MIN } t$$

$$\text{S.T.} \quad \begin{pmatrix} 1 & A_0 x + b_0 \\ A_0 x + b_0 & c_0^T x + d_0 + t \end{pmatrix} \ge 0$$

$$\begin{pmatrix} 1 & A_i x + b_i \\ A_i x - b_i & c_i^T x + d_i \end{pmatrix} \ge 0 \quad i = 1 \ldots L$$

$$F_0 = \begin{pmatrix} 1 & b_0 & & \\ b_0 & d_0 & & 0 \\ & & 1 & b_1 \\ 0 & & a_1 & d_1 & \ldots \end{pmatrix}$$

$$F_i = \begin{pmatrix} 0 & a_{0i} & & & \\ a_{0i} & c_{0i} & & 0 & \\ & & 0 & a_{1i} & \\ & 0 & a_{1i} & c_{1i} & \\ & & & & \ddots \end{pmatrix}$$

$$F_t = \begin{pmatrix} 0 & 0 & & \\ 0 & 1 & & 0 \\ & & 0 & \\ & 0 & & \end{pmatrix}$$

NOTE COMPUTE EXACT SOLUTION IS DIFFICULT

DEF

α-APPROXIMATION ALGORITHM.

x ... INSTANCE OF PROBLEM

$OPT(x)$ ... VALUE OF SOLUTION

$f(x)$ ... VALUE COMPUTED BY ALGORITHM f

f IS α-APPROXIMATION IF

$$\forall x \begin{cases} OPT(x) \le f(x) \le \alpha \cdot OPT(x) & \alpha > 1 \\ \alpha \cdot OPT(x) \le f(x) \le OPT(x) & \alpha < 1 \end{cases}$$

↳ RELATIVE PERFORMANCE GUARANTEES.

EXAMPLE: RANDOMIZED 0.5_APPROXIMATION FOR MAXCUT

$G = (V, E)$

$\forall v \in V$ PUT INTO S WITH PROBABILITY 0.5

$\Rightarrow$ $\forall e \in E$ e IN CUT $= 0.5$

X = # EDGES IN CUT

$\mathbb{E}X = \sum_e X_e = \sum_e 0.5 = 0.5 \cdot |E|$

AND

 → ALL EDGES ACROSS $\Rightarrow$

0.5 APPROXIMATION (IN EXPECTATION)

0.875 APPROXIMATION ALGORITHM
(GOEMANS – WILLIAMSON)

- INTEGER PROGRAM
- RELAXATION (NO INT. CONSTRAINT)
- EXPRESS AS SDP
- ROUND SOLUTION

$G=(V,E)$, $V=\{1,2,\dots m\}$

$\forall i \in \{1,\dots m\}$ $x_i = +1$ IF $i \in S$

$= -1$ IF $i \notin S$

$x_i x_j$ EDGE : $\dfrac{1-x_i x_j}{2} = \begin{cases} 0 \text{ .. NOT CUT} \\ 1 \text{ .. IN CUT} \end{cases}$

$$(P)\quad \begin{cases} \text{MAX} \displaystyle\sum_{\{i,j\}\in E} \frac{1-x_i x_j}{2} \\ \text{ST.} \quad x_i \in \{-1,1\} \quad i=1\dots n \end{cases}$$

OPT(P) = SIZE OF MAX CUT

• RELAXATION:

$x_i \in \{-1,1\} \to x_i \in [-1,1]$

BUT THEN TROUBLE WITH ROUNDING

$x_i \rightsquigarrow u_i \in S^{n-1} = \{u \in \mathbb{R}^n : \|u\| = 1\}$

(n-1) DIMENSIONAL UNIT SPHERE

S^1 

$$(P')\quad \begin{cases} \text{MAX} \displaystyle\sum_{(i,j)\in E} \frac{1-u_i^T u_j}{2} \\ \text{ST} \quad u_i \in S^{n-1} \end{cases}$$

NOW OPTIMIZE OVER VECTORS :-(

R $x_i = 1 \rightsquigarrow (1,0,0,\dots,0)$

$x_1 = -1 \sim u_1 = (-1, 0, 0 \ldots 0)$

$\Rightarrow \quad OPT(P) \le OPT(P')$

- WRITING AS SDP

USE SUBSTITUTION

$y_{ij} = u_i^T u_j$

$U = \left(u_1 \middle| u_2 \middle| \ldots u_n \right) \qquad Y = U^T U \Rightarrow$

$\Rightarrow Y \succeq 0$

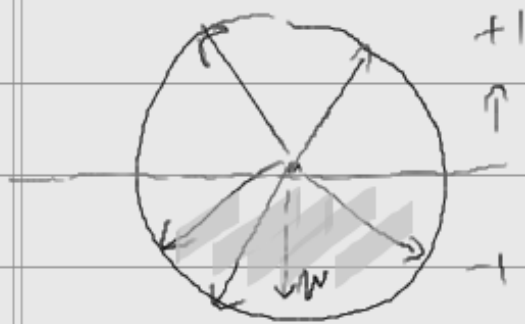
$$(SDP) \begin{cases} \text{MAX} \displaystyle\sum_{ij \in E} \frac{1 - y_{ij}}{2} = f(Y) \\ \text{ST.} \quad y_{ii} = 1 \quad \ldots\ldots \quad u_i \in S^{n-1} \\ \qquad\quad Y \succeq 0 \end{cases}$$

(FOR Y $\exists$ UNIQUE $Y = U^T U$)

NOW „SOLVE" SDP ST. $f(Y) \ge OPT(SDP) - \varepsilon \ge OPT(P) - \varepsilon$

COMPUTE M_i FROM Y .. EASY
(CHOLESKY FACTORIZATION)

ROUNDING TO $-1, 1$:



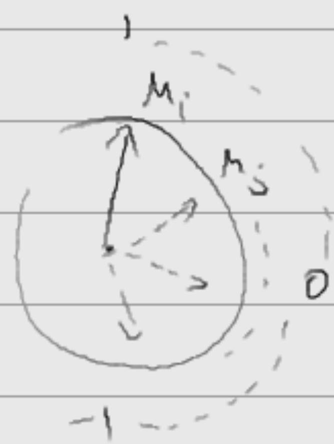
RANDOMLY PICK HALF OF S^{n-1} AND SAY THAT VECTORS IN IT ARE +1 AND OTHERS ARE -1.

CHOSE r RANDOMLY IN S^{n-1}

$$M \longrightarrow \begin{cases} 1 & \text{IF } r^T M \geq 0 \\ -1 & \text{OTHERWISE} \end{cases}$$

IS IT GOOD?

$$\frac{1 - M_i^T M_j}{2}$$



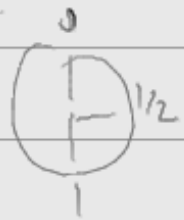
FAR $\Rightarrow$ BIG CONTRIBUTION $\Rightarrow$ LIKELY TO BE SEPARATED BY r

LEMMA:

LET $u, u' \in S^{n-1}$. PROBABILITY THAT u AND u' ARE MAPPED INTO DIFFERENT VALUES IS

$$\frac{1}{\pi} \arccos u^T u'$$

→ TEST



GETTING THE BOUND:

WE WANT $\mathbb{E}\left(\sum_{ij \in E} \frac{\arccos u_i^T u_j}{\pi} \right)$

BUT WE KNOW ONLY $\frac{1 - u_i u_j}{2}$

LEMMA

$$\frac{\arccos z}{\pi} \geq 0.878\ldots \frac{1-z}{2}$$

[BY MINIMIZING $2 \cdot \frac{\arccos z}{\pi (1-z)}$ ON $[-1, 1]$]

SOLVING SDP – INTERIOR POINTS METHOD

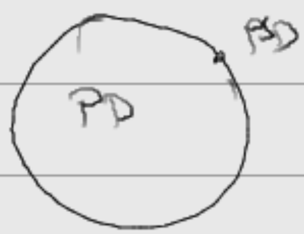
$$(SDP)\quad \begin{cases} \max\ Tr(C^T X) \\ \text{S.T.}\quad A(X) = b \\ \qquad\quad X \succeq 0 \end{cases}$$

$Tr(A_1^T X) = b_1$

$\vdots$

$Tr(A_n^T X) = b_n$

DROP $X \succeq 0$ AND USE $X \succ 0$ INSTEAD & ADD SOME PENALTY FOR X CLOSE TO BE NOT POS. DEF:



REFORMULATION:

$$(P_\mu)\quad \begin{cases} \max\ f_\mu(X) = Tr(C^T X) + \mu \ln \det X \\ \text{S.T.}\quad A(X) = b \\ \qquad\quad X \succ 0 \end{cases}$$

IN SOME SENSE „INTERIOR"

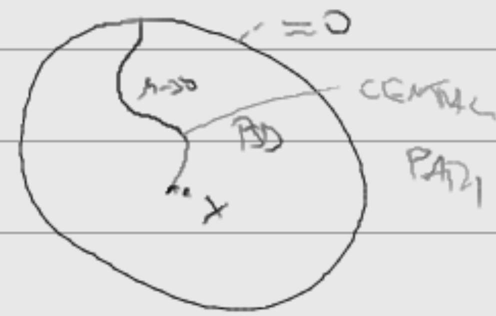
$X \succeq 0$ & NOT $X \succ 0 \Rightarrow$ EIGENVALUE $= 0 \Rightarrow$

det = 0.

„ GOING WITH X TO BOUNDARY IS PENALIZED"

X_μ^* – SOLUTION OF (P_μ)

$\lim_{\mu \to 0} X_\mu^* = X^*$



WANT:

- X_μ^* IS UNIQUE (LIMIT IS BETTER, NICER)
- GET NECESSARY EQUATIONS & INEQUALITIES THAT X_μ^* MUST SATISFY (SOLVING SYSTEM)
- SHOW THAT THEY COMPLETELY DESCRIBE X_μ^*

UNIQUENESS

LEMMA:

FUNCTION $X \to \ln \det X$ IS STRICTLY CONCAVE OVER $X \succ 0$

$\mathrm{Tr}(C^T X)$ .. LINEAR + STRICTLY CONCAVE $\Rightarrow$ UNIQUE

$\text{Max } f_\mu(X)$

S.T. $g_1(X) = 0$

$\vdots$

$g_m(X) = 0$

$X \succ 0$

$\text{MIN } -f_\mu(X)$

S.T. $g_1(X) \le 0$

$\vdots$

$g_m(X) \le 0$

$X \succ 0$

CONVEX PROGRAM → KKT, $\exists\, X^*$

1) $\lambda_1 \dots \lambda_m \ge 0$

2) $\lambda_i\, g_i(X^*) = 0$

3) $\nabla f(X^*) + \sum \lambda_i \nabla g_i(X^*) = 0$

1) $\lambda_1 \dots \lambda_m \in \mathbb{R}$

2) $g_i(X^*) = 0$

3) $\nabla f(X^*) + \sum \lambda_i \nabla g_i(X^*) = 0$

$\left.\begin{array}{l} g_i(X) \le 0 \\ -g_i(X) \le 0 \end{array}\right\} = g_i(X^*) = 0$

MIMICKING EQUALITY

$\lambda_1 \dots \lambda_m$ .. LAGRANGE MULTIPLIERS

LEMMA: $\nabla \ln\det(X) = (X^T)^{-1}$; $\nabla \operatorname{Tr}(C^T X) = C^T$

$\left(\Rightarrow \nabla f_\mu(X) = C^T + \mu (X^T)^{-1} \right)$

$g_i(X) = \operatorname{Tr}(A^T X) \Rightarrow \nabla g_i(X) = A^T$

$\Rightarrow$

$$C^T + \mu (X^T)^{-1} + \sum_{i=1}^{m} y_i A_i^T = 0$$

$$A(X) = 0$$

$\rightarrow$ SOLVES ONLY FOR $X \in \mathbb{R}^{n \times n}$, NOT $X \succ 0$

CAN BE SOLVED BY NEWTONS METHOD:

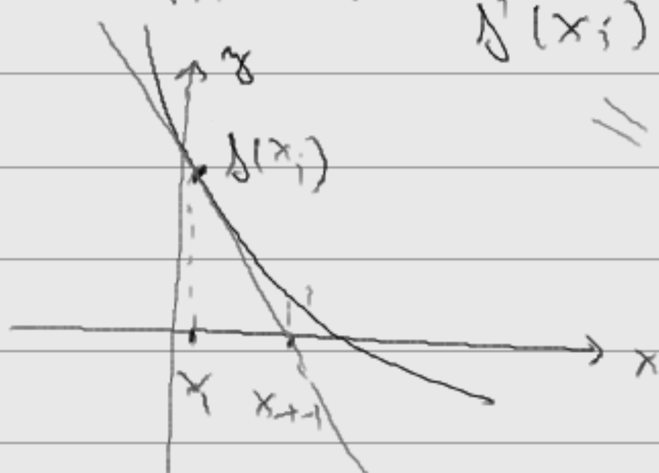
$f: \mathbb{R} \to \mathbb{R}$

FIND X ST $f(X) = 0$

START WITH $X_0 \in \mathbb{R}$

$$X_{i+1} = X_i - \frac{f(x_i)}{f'(x_i)}$$

$\Rightarrow \Delta X$



$$y = f(x_i) + f'(x_i)(x - x_i)$$

$y = 0, \; x = x_{i+1}$

FORMULA FOR ΔX CAN BE OBTAINED.
(GIVEN $\Delta X \in SYM_n$)

START AT X_0

- COMPUTE μ (BASED ON X_i)
- COMPUTE ΔX
- $X_{i+1} = X_i + \Delta X$

ITERATE

NOTE THIS PROCESS IS ALSO SOLVING DUAL AT THE SAME TIME → KNOW WHEN TO STOP ITERATING.