MATH213 HW 5

due **Mar 7** before class

1: How many one-to-one functions are there from the set A of five elements to set B of size:
a) 4
b) 5
c) 6
d) 7
($f : A \to B$, f is one to one, $|A| = 5$ and $|B|$ varies)

2: Show that if there are 30 students in class then there are at least two whose name starts with the same letter.

3: Show that among any group of five (not necessarily consecutive) integers are two whose remainder after dividing by 4 is the same.

4: What is the minimum number of students, each of whom comes from one of the 50 states, who must be enrolled in the university to guarantee that there are at least 100 who come from the same state.

5: Let (x_i, y_i), $i = 1, 2, 3, 4, 5$ be set of five distinct point in the plane with integer coordinates (x_i, y_i). They define 20 lines which have these points as endpoints. Show that least one of these lines have midpoint with integer coordinates.
Example: line joining points (x_1, y_1) and (x_2, y_2) has midpoint $(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2})$.

6: a) List all 2-permutations of $S = \{1, 2, 3, 4\}$
b) List all 3-combinations of $S = \{1, 2, 3, 4\}$

7: Compute the following (final result is just integer):
a) $C(6, 3)$ b) $C(8, 1)$ c) $C(8, 8)$ d) $C(10, 9)$

8: How many terms are there in expansion of $(x + y)^{100}$ after like terms are collated?

9: Let $n \geq r \geq k$. Prove the following identity

$$\binom{n}{r}\binom{r}{k} = \binom{n}{k}\binom{n-k}{r-k}$$

(a) by a combinatorial argument
(b) using algebraic proof - by writing binomial coefficients using factorials.

10: One hundred tickets, numbered $1, 2, \ldots, 100$ are sold to 100 different people for a drawing. There is a grand prize and three other smaller equal prizes. How many ways are there to award the prizes if
(a) there are no restrictions?
(b) the person holding ticket 1 wins one of the prizes?
(c) the people holding tickets 1 and 2 both win some prizes?
(d) the grand prize winner is a person holding ticket $1, 2$ or 3?

11: Find
(a) the coefficient of x^3y^7 in $(2x+y)^{10}$;
(b) the coefficient of $x^{13}y^{77}$ in $(3x-2y)^{90}$;
(c) a simpler formula for $\sum_{k=0}^{10}\binom{20}{k}\binom{15}{10-k}$.

12: Prove that
(a) $\sum_{k=0}^{n}\binom{n}{k}(-1)^k\,3^{n-k}2^k = 1$ for each positive integer n;
(b) $n\,3^{n-1} = \sum_{k=1}^{n}\binom{n}{k}k2^{n-k}$ for each positive integer n. *(Hint: Use derivative of binomial theorem)*

13: Find one binomial coefficient equal to the following expression

$$\binom{n}{k} + 3\binom{n}{k-1} + 3\binom{n}{k-2} + \binom{n}{k-3}.$$

(Hint: Use identities.)