MATH413 HW 7

due **Apr 6** before class

1: Find generating functions for the following sequences:
(a) $0, 0, 0, 6, -6, 6, -6, 6, -6, 6, -6, \ldots$
(b) $1, 2, 4, 1, 3, 9, 1, 4, 16, 1, 5, 25, 1, 6, 36, \ldots$
(c) $\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \frac{5}{6}, \frac{6}{7}, \frac{7}{8}, \ldots$

2: For every $n \geq 0$ determine the coefficient at x^n in $(1+x)^2(1+x^2)^2(1+x^4)^2(1+x^8)^2 \cdots$ which is equal to $\prod_{i=0}^{\infty}(1+x^{2^i})^2$.

3: *P. 258, #9* Let h_n equal the number of different ways in which the squares of a 1-by-n chessboard can be colored, using the colors red, white, and blue so that no two squares that are colored red are adjacent. Find and verify a recurrence relation that h_n satisfies. Then find a formula for h_n.

4: *P. 258, #11* The *Lucas numbers* $l_0, l_1, l_2, \ldots, ln, \ldots$ are defined using the same recurrence relation defining the Fibonacci numbers, but with different initial conditions:

$$l_n = l_{n-1} + l_{n-2}, (n \geq 2), l_0 = 2, l_1 = 1.$$

Prove that
(a) $l_n = f_{n-1} + f_{n+1}$ for $n \geq 1$
(b) $l_0^2 + l_1^2 + \ldots + l_n^2 = l_n l_{n+1} + 2$ for $n \geq 0$.

5: *P.260, #22* Determine the exponential generating function for the sequence of factorials

$$0!, 1!, 2!, 3!, \ldots, n!, \ldots$$

6: *P.260, # 24* Let S denote the multiset $\{\infty \cdot e_1, \infty \cdot e_2, \ldots, \infty \cdot e_k\}$. Determine the exponential generating function for the sequence $h_0, h_1, h_2, \ldots, h_n, \ldots$, where $h_0 = 1$ and for $n \geq 1$,
(b) h_n equals the number of n-permutations of S in which each object occurs at least four times.

(c) h_n equals the number of n-permutations of S in which e_1 occurs at least once, e_2 occurs at least twice, ..., e_k occurs at least k times.

7: *P.260, # 26* Determine the number of ways to color squares of a 1-by-n chessboard using the colors red, blue, green, and orange if an even number of squares is to be colored red and an even number is to be colored green.