MATH413 HW 10

due **Apr 25 strictly before class**

Solutions without explanation will receive no points.

1: *P.320, # 22(a)* Compute p_6 (the partition number of 6) and construct a Hasse diagram of partialy ordered set $\mathcal{P}_6$ where $\mathcal{P}_n$ contains all partitions of n (in our case $n = 6$). Suppose that $a, b \in \mathcal{P}_n$ and

$$a : n = a_1 + a_2 + \cdots + a_n \text{ where } a_1 \geq a_2 \geq \cdots \geq a_n \geq 0$$

and

$$b : n = b_1 + b_2 + \cdots + b_n \text{ where } b_1 \geq b_2 \geq \cdots \geq b_n \geq 0.$$

Notice that we allow partitions that include 0. That is because it is easier to write formaly the following. We say that $a \leq b$ if

$$\forall i = \{1, \cdots, n\} : a_1 + \cdots + a_i \leq b_1 + \cdots + b_i.$$

So we have relation $\leq$ on $\mathcal{P}_6$ and it allows us to draw a partially ordered set of the relation. See Page 296 for more details abot $\mathcal{P}_6$.

2: *P.318, #26* Determine the conjugate partition of each of the following patitions:

(a) 12 = 5+4+2+1

(b) 15 = 6+4+3+1+1

(c) 20 = 6+6+4+4

(d) 21 = 6+5+4+3+2+1

3: *P.318, #27* For each integer $n > 2$, determine a self-conjugate partition of n that has at least two parts.

4: *P.318, #30* Prove that the partition function p_n (=number of partitions of n) satisfies

$$p_{n+1} > p_n$$

for $n \geq 2$.

5: Prove that the number of partitions of n in which no part appears exactly once is equal to the number of partitions of n with no parts congruent to 1 or 5 (mod 6).

6: By considering partitions with distinct (that is, non-repeated) parts, prove that

$$\prod_{k=1}^{\infty}(1+x^k) = 1 + \sum_{m=1}^{\infty} \frac{x^{m(m+1)/2}}{\prod_{k=1}^{m}(1-x^k)}$$

(Hint: look for a "maximal triangle" rather than a maximal square (Durfee square) in the Ferrers diagram).

7: Using the difference sequence method, find a closed form the following sum:

$$\sum_{k=0}^{n} k^4 - k.$$

8: *P.316, #7* The general term h_n of a sequence is a polynomial in n of degree 3. If the first four entries in the 0^{th} row of its difference table are 1,-1,3,10, determine h_n and a formula for $\sum_{k=0}^{n} h_k$.