

INTERIOR-POINT METHODS (II.1)

(LOQD, VANDENBERGHE - CONVEX OPTIMIZATION)

$$(P) \begin{cases} \min f(x) \\ \text{s.t. } g_i(x) \leq 0 \quad i=1 \dots m \end{cases}$$

f, g CONVEX TWICE CONT. DIFFERENTIABLE
 x^* OPT. SOLUTION EXISTS

(P) SUPERCONSISTENT (STRICTLY FEASIBLE)

$$\exists x \quad g_i(x) < 0 \quad i=1 \dots m$$

KKT THEOREM $\Leftrightarrow \lambda^* \in \mathbb{R}^m$ s.t.

$$1) \lambda_i^* \geq 0 \quad \forall i$$

$$2) \lambda_i^* g_i(x^*) = 0 \quad \forall i$$

$$3) \nabla f(x^*) + \sum \lambda_i^* \nabla g_i(x^*) = 0$$

COVERS

LP, QP, QLOP, GP

(SDP IF ONE TRIS)

(11.2) LOGARITHMIC BARRIER

IDEA:

CHANGE (P) TO m

$$\text{MIN } f(x) + \sum_{i=1}^m I(g_i(x))$$

WHERE

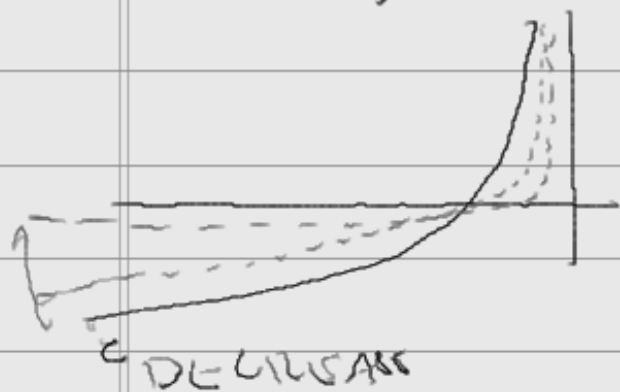
$$I(m) = \begin{cases} 0 & \text{IF } m \leq 0 \\ +\infty & \text{IF } m > 0 \end{cases}$$

INDICATOR FUNCTION FOR \mathbb{R}

(LIKE "TOTAL" PENALTY FUNCTION)

APPROXIMATION:

$$I(m) \approx -c \cdot \log(-m) \quad c > 0$$



APPROXIMATION IMPROVES

AS $c \rightarrow 0$ DIFFICULT TO SOLVE

$\epsilon \rightarrow +\infty$ IS BETTER APPROX. WITH HARD.

$$\Rightarrow \text{MIN } f(x) - \frac{1}{\epsilon} \sum \log(-g_i(x))$$

\rightarrow IN ∞ CONSTRAINTS, SMOOTH APPROXIMATION OF ✓

DEF: LOGARITHMIC BARRIER FUNCTION

$$\phi(x) = - \sum_{i=1}^m \log(-g_i(x))$$

$\forall x \text{ s.t. } g_i(x) < 0 \quad \leftarrow \text{INTERIOR OF FEASIBLE SOLUTIONS}$

$\phi(x)$ IS CONVEX SINCE $-\log$ IS CONVEX
STRUCTURE BECAUSE Δg_i CONVEX

NOTE:

$$\nabla \phi(x) = \sum \frac{1}{-g_i(x)} \nabla g_i(x)$$

$$H \phi(x) = \sum \frac{1}{g_i(x)^2} \nabla g_i(x) \nabla g_i(x)^T + \nabla^2 \phi(x) + \sum \frac{1}{-g_i(x)} H g_i(x)$$

FOR $\epsilon > 0$ DEFINE $x^\epsilon(t)$ AS SOLUTION TO

$$\min_x \epsilon f(x) + \phi(x)$$

{SUPPOSE ALWAYS UNIQUE}

CENTRAL PATH: $\{x^\epsilon(t) \mid \epsilon > 0\}$

so for $x^*(t)$:

$$0 = t \nabla f(x^*(t)) + \nabla \phi(x^*(t))$$

$$0 = t \nabla f(x^*(t)) + \sum_{i=1}^m \frac{1}{-g_i(x^*(t))} \nabla g_i(x^*(t))$$

EX FOR LINEAR PROGRAMMING

$$\text{(LP)} \begin{cases} \text{MIN } c^T x & \text{MIN } t c^T x + \phi(x) \\ \text{s.t. } Ax \leq b & \end{cases} \quad A \in \mathbb{R}^{m \times n}$$
$$\phi(x) = - \sum_{i=1}^m \log(b_i - a_i^T x)$$

WHERE a_i^T IS ROW OF A

$$\nabla \phi(x) = \sum_{i=1}^m \frac{1}{b_i - a_i^T x} a_i$$

$$\left[\begin{aligned} H\phi(x) &= \sum_{i=1}^m \frac{1}{(b_i - a_i^T x)^2} a_i a_i^T \\ \text{JUSTIFIED THAT JUST } \nabla = 0 \text{ IS FINE} \end{aligned} \right]$$

$$0 = t c + \sum_{i=1}^m \frac{1}{b_i - a_i^T x} a_i$$

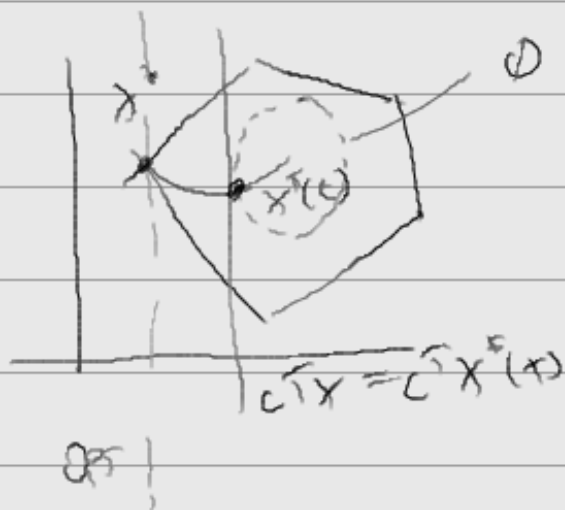
$$t c = -\nabla \phi(x) \quad \leftarrow \text{PARALLEL TO } C$$

BONUS: $\nabla \phi(x^*(t))$ IS \perp TO LEVEL

CURVE OF $\phi(x) = \phi(x^*) \Rightarrow$

$c^T x = c^T x^*(t)$ IS A TANGENT OF LEVEL

CURVES FOR ϕ



DUAL POINTS FROM CENTRAL PATH

$x^*(t)$ GIVES POINT IN DUAL:

RECALL:

$$(P) \begin{cases} \min & J(x) \\ \text{s.t.} & g_i(x) \leq 0 \end{cases}$$

$$(DP) \begin{cases} \max_{\lambda} & h(\lambda) = \inf_x L(x, \lambda) = J(x) + \sum \lambda_i g_i \\ & \lambda \geq 0 \end{cases}$$

λ FEASIBLE IF $h(\lambda) \neq -\infty$

$$\lambda_i = \frac{-1}{\epsilon g_i(x^*(t))}$$

$$1, \lambda > 0 \text{ since } \sum_{i=1}^m \lambda_i \leq 0$$

GRADIENT CONDITION

POSITIONS

$$2, 0 = \epsilon \nabla f(x^*(t)) + \sum_{i=1}^m \frac{1}{-g_i(x^*(t))} \nabla g_i(x^*(t))$$

$$0 = \nabla f(x^*(t)) + \sum \lambda_i \nabla g_i(x^*(t))$$

HENCE $x^*(t)$ MINIMIZES

$$L(x, \lambda) = f(x) + \sum \lambda_i g_i(x)$$

$\Rightarrow \lambda$ IS FEASIBLE. - CALL IF $\lambda^*(t)$

$$h(\lambda^*(t)) = f(x^*(t)) + \sum_{i=1}^m \lambda^*(t)_i g_i(x^*(t))$$

$$= f(x^*(t)) - m/\epsilon$$

\Rightarrow DUALITY GAP m/ϵ AS

$$f(x^*(t)) - h(\lambda^*(t)) \leq m/\epsilon$$

AS $t \rightarrow \infty$ BY $x^*(t) \rightarrow x^*$

USING KKT CONDITIONS

x IS OPT SOLUTION $x^*(t)$ IFF $\exists \lambda$

1) $g_i(x) \leq 0 \quad \forall i$

2) $\lambda \geq 0$

3) $\nabla f(x) + \sum \lambda_i \nabla g_i(x) = 0$

4) $-\lambda_i g_i(x) = 1/t$

RECALL IN KKT $\lambda_i g_i(x) = 0$

ϵ MAKES DEFORMATION OF KKT CONDITIONS