SEMIDEFINITE PROGRAMMING

DEFINITION

LET $A \in \mathbb{R}^{m \times m}$. THE TRACE OF A IS

$$tr(A) = \sum_{i=1}^{m} a_{ii}$$

$$SYM_m = \{ X \in \mathbb{R}^{m \times m} : x_{ij} = x_{ji} \ \forall i,j \}$$

SYMMETRIC MATRICES $m \times m$

RECALL (LP)

$$(LP) \begin{cases} \max & c^T x \\ \text{s.t.} & Ax = b \\ & x \geq 0 \end{cases}$$

$x = (x_1 \dots x_n)$, $c \in \mathbb{R}^n$, $b \in \mathbb{R}^m$, $A \in \mathbb{R}^{m \times n}$

$$(LP)\begin{cases} \text{MAX} & C \cdot X \\ \text{S.T.} & a_1 \cdot X = b_1 \\ & a_2 \cdot X = b_2 \\ & \vdots \\ & a_m \cdot X = b_m \\ & X \geq 0 \end{cases}$$

a_i .. i-TH ROW OF A

$c, x, a_1 \ldots a_m$ ARE VECTORS $\in \mathbb{R}^n$

$b_1, b_2, \ldots b_m \in \mathbb{R}$

REPLACE • VECTOR SPACE $\mathbb{R}^n$ BY VECTOR SPACE

SYM_n

- THE DOT PRODUCT BY

$$X \cdot Y = \langle X|Y \rangle = T_r(X^T Y) =$$
$$= \sum_{i=1}^{n} \sum_{j=1}^{n} x_{ij} y_{ij}$$

- $x \geq 0$ BY X IS POSITIVE SEMIDEFINITE. DENOTED BY $X \succeq 0$

($X \succ 0$, X IS POSITIVE DEFINITE)

$$(SDP)\begin{cases} \max \ \mathrm{Tr}(C^T X) \\ \text{ST.} \ \ \mathrm{Tr}(A_1^T X) = b_1 \\ \qquad \vdots \\ \qquad \mathrm{Tr}(A_m^T X) = b_m \\ \qquad X \succeq 0 \end{cases}$$

$C, A_1 \dots A_m, X \in SYM_m$

$b_1 \dots b_m \in \mathbb{R}$

$$\mathrm{Tr}\left(\begin{pmatrix} C_{11} & C_{12} \\ C_{12} & C_{22} \end{pmatrix}^T \begin{pmatrix} X_{11} & X_{12} \\ X_{12} & X_{22} \end{pmatrix}\right) = C_{11}X_{11} + 2C_{12}X_{12} + C_{22}X_{22}$$

→ EQUIVALENT FORM OF (SDP)

$$(SDP)\begin{cases} \max \ \sum_{i,j} c_{ij} X_{ij} \\ \text{ST.} \ \ \sum_{ij} a_{ijk} X_{ij} = b_k, \quad k = 1 \dots m \\ \qquad X \succeq 0 \end{cases}$$

(LP) IS SPECIAL CASE OF (SDP)

$$c \longrightarrow \begin{pmatrix} c_1 & & & 0 \\ & c_2 & & \\ & & \ddots & \\ 0 & & & c_n \end{pmatrix} = C$$

$$a_i \longrightarrow \begin{pmatrix} a_{i1} & & & 0 \\ & a_{i2} & & \\ & & \ddots & \\ 0 & & & a_{in} \end{pmatrix} = A_i$$

$$x \longrightarrow \begin{pmatrix} x_1 & & & 0 \\ & x_2 & & \\ & & \ddots & \\ 0 & & & x_n \end{pmatrix} = X$$

NOTE: $x \geq 0 \Longleftrightarrow X \succeq 0$

(x_i ARE EIGENVALUES OF X)

NOTE HOW TO HANDLE INEQUALITY?

$$A_i \bullet X \geq b_i$$

EXTEND X:

$$\begin{pmatrix} A_i & 0 \\ 0 & -1 \end{pmatrix} \cdot \begin{pmatrix} X & 0 \\ 0 & x_i' \end{pmatrix} = b_i$$

AS x_i' IS ANYTHING ≥ 0

DUAL FORM:

(DSDP)
$$\begin{cases} \text{MINIMIZE} \quad b^T \cdot y \\ \text{ST.} \\ \quad y_1 \cdot A_1 + y_2 A_2 + \dots y_n A_n - C \succeq 0 \end{cases}$$

DEF:

- (SDP) IS STRICTLY FEASIBLE IF $\exists$ FEASIBLE X WHICH IS POSITIVE DEFINITE ($X \succ 0$)
- (DSDP) IS STRICTLY FEASIBLE IF $\exists$ FEASIBLE y ST. ($y \cdot A - C \succ 0$)

THEOREM (STRONG DUALITY OF SDP)

- IF (SDP) IS STRICTLY FEASIBLE AND HAS OPT. SOLUTION OF VALUE γ THEN (DSDP) IS FEASIBLE AND HAS OPT. SOLUTION OF VALUE γ
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THEOREM (SOLVABLE IN PTIME)

LET (SDP) BE FEASIBLE, FEASIBLE REGION F OF (SDP) BE BOUNDED, LET $R \in \mathbb{N}$ BE SUCH THAT $R \geq \sqrt{T_r(X^T X)}$ $\forall X \in F$ ($\|X\|_2 = \sqrt{X \cdot X}$) AND $\varepsilon > 0$. LET m BE SIZE OF (SDP) [BINARY ENCODED INPUT] THEN IN PTIME WE CAN COMPUTE X' OF VALUE OPTIMUM $- \varepsilon$.

EXAMPLE .. (SDP) WHERE $\boxed{\text{ALL}}$ FEASIBLE VECTORS ARE $\boxed{\text{HUGE}}$

(DSDP):

$$X = \begin{pmatrix} 1 & 2 & & & & & & & \\ 2 & x_1 & & & & & & & \\ & & 1 & x_1 & & & & & \\ & & x_1 & x_2 & & & & & \\ & & & & 1 & x_2 & & & \\ & & & & x_2 & x_3 & & & \\ & & & & & & \ddots & & \\ & & & & & & & 1 & x_{n-1} \\ & & & & & & & x_{n-1} & x_n \end{pmatrix} \succeq 0$$

$$x_1 \begin{pmatrix} 0 & 0 & & & & \\ 0 & 1 & 0 & & & 0 \\ & 0 & 0 & 1 & & \\ & & 1 & 0 & & \\ & & & & & \\ & 0 & & & & \end{pmatrix} + x_2 \ldots + \begin{pmatrix} 1 & 2 & & & \\ 2 & 0 & & & \\ & & 1 & & \\ & & & 0 & \\ & & & & 1 \\ & & & & & \ddots \end{pmatrix}$$

IF $X \succeq$ THEN

$$\begin{pmatrix} 1 & x_i \\ x_i & x_{i+1} \end{pmatrix} \succeq 0 \Rightarrow$$

$$\Rightarrow \begin{vmatrix} 1 & x_{i-1} \\ x_{i-1} & x_i \end{vmatrix} \geq 0 \Rightarrow$$

$$x_i - x_{i-1}^2 \geq 0$$

$$x_1 \geq 4 = 2^2$$

$$x_2 \geq x_1^2 = \left(\left(\left(2^2\right)^2\right)^2\right)^m = 2^{2^m}$$

$\Rightarrow \log_2 2^{2^2} = 2^m \rightarrow$ JUST WRITING SOLUTION REQUIRES EXP. TIME !

EXAMPLE

$$\text{MINIMIZE } \frac{(c^T x)^2}{d^T x}$$

$$\text{S.T. } Ax + b \geq 0$$

WHERE $d^T x > 0$ WHENEVER $Ax + b \geq 0$

$x, c, d \in \mathbb{R}^n \quad A \in \mathbb{R}^{m \times n} \quad b \in \mathbb{R}^m$

WRITE AS (DSDP)

1) LINEAR OBJECTIVE FUNCTION

MINIMIZE t

S.T. $Ax+b \ge 0$

$$\frac{(c^T x)^2}{d^T x} \le t$$

$$(c^T x)^2 \le d^T x \cdot t$$

$$d^T x \cdot t - (c^T x)^2 \ge 0$$

$$\begin{vmatrix} t & c^T x \\ c^T x & d^T x \end{vmatrix} \ge 0$$

MIN t

$$\begin{pmatrix} a_1 \cdot x + b_1 & & & & 0 \\ & a_2 \cdot x + b_2 & & & \\ & & \ddots & & \\ & & & a_m x + b_m & \\ 0 & & & & \begin{matrix} t & c^T x \\ c^T x & d^T x \end{matrix} \end{pmatrix} \ge 0$$

EXAMPLE: QUADRATIC CONSTRAINT

$$f(x) = (Ax+b)^T(Ax+b) - c^Tx - d \le 0$$

$A \equiv [a_1\ a_2 \dots a_k]$

$c, x \in \mathbb{R}^k,\ b, d \in \mathbb{R},$

DUAL FORM

$$F(x) = F_0 + x_1 F_1 + \dots + x_k F_k$$

$$F_0 = \begin{pmatrix} 1 & b \\ b & d \end{pmatrix},\quad F_i = \begin{pmatrix} 0 & a_i \\ a_i & c_i \end{pmatrix}$$

$$f(x) = \begin{pmatrix} 1 & b + \sum x_i a_i \\ b + \sum x_i a_i & d + \sum c_i x_i \end{pmatrix} \ge 0$$

$$(d + \sum c_i x_i) - (b + \sum x_i a_i)^2 \ge 0$$

det(F(x))

$$\min f_0(x)$$
$$\text{ST} \quad f_1(x) \le 0$$
$$f_2(x) \le 0$$
$$f_L(x) \le 0$$

$$f_i(x) = (A_i x + b_i)^2 - c_i^T x - d_i$$

$$f_0(x) \le t \quad \& \quad \min t$$

$$(A_0 x + b_0)^2 - c_0^T x - d_0 - t \le 0$$

$$\min t$$

$$\text{S.T.} \quad \begin{pmatrix} 1 & A_0 x + b_0 \\ A_0 x + b_0 & c_0^T x + d_0 + t \end{pmatrix} \ge 0$$

$$\begin{pmatrix} 1 & A_i x + b_i \\ A_i x + b_i & c_i^T x + d_i \end{pmatrix} \ge 0 \quad i = 1 \ldots L$$

$$F_0 = \begin{pmatrix} 1 & b_0 & & \\ b_0 & d_0 & & 0 \\ & & 1 & b_1 \\ 0 & & a_1 & d_1 & \ldots \end{pmatrix}$$

$$F_i = \begin{pmatrix} 0 & a_{0i} & & & \\ a_{0i} & c_{01} & & & \\ & & 0 & a_{1i} & \\ & & a_{1i} & c_{1i} & \\ & & & & \ddots \end{pmatrix}$$

$$F_t = \begin{pmatrix} 0 & 0 & & \\ 0 & 1 & & \\ & & 0 & \\ & & & \ddots \end{pmatrix}$$

MAXCUT USING SDP

DEF: GRAPH

$G = (V, E)$

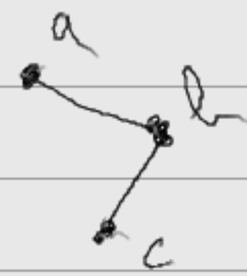
V ... VERTICES — JUST SOME SET

$E \subseteq \binom{V}{2}$ .. EDGES

EX

$V = \{a, b, c\}$

$E = \{\{a,b\}, \{b,c\}\}$

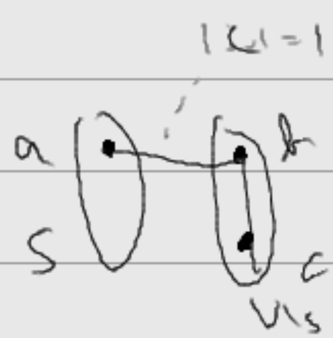


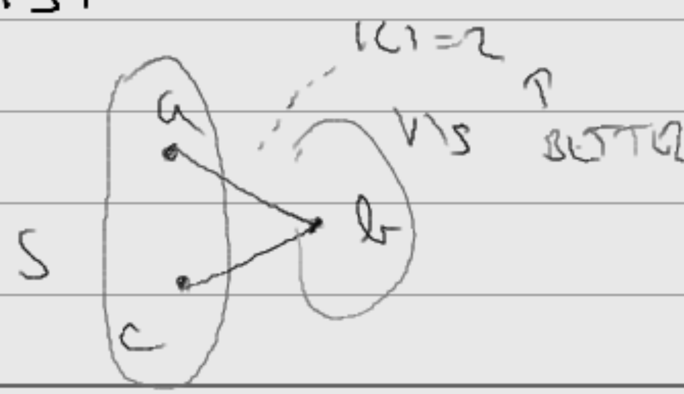
MAXCUT PROBLEM

INPUT: GRAPH G

OUTPUT $S \subseteq V$ S.T.

$C = |\{e \in E : |e \cap S| = 1\}|$ IS MAXIMIZED





NOTE COMPUTE EXACT SOLUTION IS NP-COMPLETE

(MEANS NOT FAST)

DEF

α-APPROXIMATION ALGORITHM.

x ... INSTANCE OF PROBLEM

$OPT(x)$ ... VALUE OF SOLUTION

$f(x)$ ... VALUE COMPUTED BY ALGORITHM f

f IS α-APPROXIMATION IF

FOR MIN

$$\forall x \begin{cases} OPT(x) \le f(x) \le \alpha \cdot OPT(x) & \alpha > 1 \\ \alpha \cdot OPT(x) \le f(x) \le OPT(x) & \alpha < 1 \end{cases}$$

FOR MAX

↳ RELATIVE PERFORMANCE GUARANTEES.

EXAMPLE: RANDOMIZED 0.5_APPROXIMATION FOR MAXCUT

$G = (V, E)$

$\forall v \in V$ PUT INTO S WITH PROBABILITY 0.5

$\forall e \in E$ e IN CUT WITH PROB. 0.5

X = # EDGES IN CUT

$$\mathbb{E}X = \sum X_e = \sum 0.5 = 0.5 \cdot |E|$$

GIVES 0.5 APPROXIMATION (IN EXPECTATION)

- EXISTS ALSO DETERMINISTIC VERSION
- $0.5(1+\frac{1}{m})$ APPROX KNOWN WHERE $m=|E|$

0.878 APPROXIMATION ALGORITHM
(GOEMANS – WILLIAMSON) 1995

$$\left.\begin{array}{l}\text{– INTEGER PROGRAM (P)}\\ \text{– RELAXATION (NO INT. CONSTRAINT)}\\ \text{– EXPRESS AS SDP}\\ \text{– ROUND SOLUTION}\end{array}\right\}\ \text{PLAN OF ATTACK}$$

$G=(V,E)$, $V=\{1,2,\dots m\}$

$\forall i\in\{1,\dots m\}$ $\quad x_i = +1$ IF $i\in S$

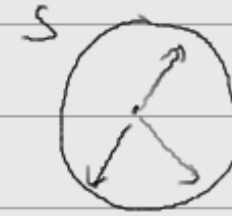
$\qquad\qquad\qquad\qquad = -1$ IF $i\notin S$

$x_i x_j$ EDGE : $\dfrac{1-x_ix_j}{2} = \begin{cases} 0 \ \dots \text{NOT IN CUT} \\ 1 \ \dots \text{IN CUT}\end{cases}$

$$(P)\ \begin{cases} \max \sum_{ij\in E} \frac{1-x_ix_j}{2} \\ \text{S.T.} \quad x_i \in \{1,-1\} \qquad i=1\dots \underset{|V|}{\overset{}{\underset{\|}{n}}}\end{cases}$$

OPT(P) = SIZE OF MAX CUT

- RELAXATION

$x_i \in \{-1, 1\} \to x_i \in [-1, 1]$ 

BUT THEN TROUBLE WITH ROUNDING

$x_i \rightsquigarrow u_i \in S^{n-1} = \{u \in \mathbb{R}^n : \|u\| = 1\}$

(n-1) DIMENSIONAL UNIT SPHERE

$$(P') \begin{cases} \text{MAX} \sum_{ij \in E} (1 - u_i^T u_j)/2 \\ \text{S.T.} \quad u_i \in S^{n-1} \end{cases}$$

↳ OPTIMIZING OVER VECTORS.. (VECTOR PROGRAM)

OPT (P) ≤ OPT (P') : FOR SOLUTION $x_1 \ldots x_n$:

$$u_i = \begin{cases} (1, 0, 0 \ldots 0) & \text{IF } x_i = 1 \\ (-1, 0, 0 \ldots 0) & \text{IF } x_i = -1 \end{cases}$$

- REWRITING TO SDP

SAY $y_{ij} = u_i^T u_j$ ---- E

$U = (u_1 | u_2 | \ldots | u_n)$ THEN $Y = U^T U$

RECALL LINEAR ALGEBRA:

Y POSITIVE SEMIDEFINITE IFF $\exists$ U ST $U^T U = Y$

[U FROM Y BY CHOLESKY FACTORIZATION]

$$
(SDP)\begin{cases} \text{MAX } \sum_{ij \in E} (1-y_{ij})/2 = f(Y) \\ \text{S.T. } \quad y_{ii} = 1 \ \ldots \ \|u_i\| = 1 \\ \qquad Y \succeq 0 \end{cases}
$$

NOW "SOLVE" SDP ST. $f(Y) \geq OPT(SDP) - \varepsilon \geq OPT(P) - \varepsilon$

(SDP CAN BE SOLVED UP TO ε FOR ANY $\varepsilon > 0$)

• ROUNDING TO $(-1,1)$

RANDOMLY PICK HALF OF S^{n-1} AND SAY THAT VECTORS IN IT ARE +1 AND OTHERS ARE -1.

FORMALLY: PICK $w \in S^{n-1}$ RANDOMLY

$$
u \longrightarrow \begin{cases} 1 & \text{IF } w^T u \geq 0 \\ -1 & \text{OTHERWISE} \end{cases}
$$

IS IT GOOD? INTUITION:

$(1 - u_i^T u_j)/2$ CONTRIBUTES

A LOT LIKELY TO BE SEPARATED

AND LITTLE WE DON'T CARE MUCH

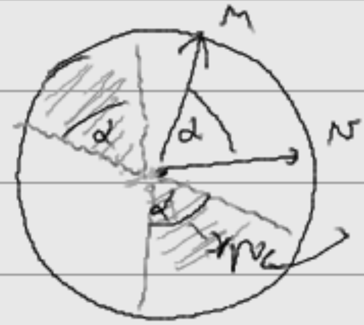
LEMMA:

LET $u, u' \in S^{n-1}$. PROBABILITY THAT u AND u' ARE MAPPED INTO DIFFERENT VALUES IS $\frac{1}{\pi} \arccos u^T u'$.

PROOF

$\cos\alpha = u^T u' \Rightarrow \alpha = \arccos u^T u'$

PROJECTION



SEPARATING $\frac{2\alpha}{2\pi}$

$\square$

WANT: $E\left(\sum \frac{1}{\pi} \arccos u_i^T u_j\right)$, HAVE $\sum (1 - u_i^T u_j)/2$

LEMMA:

$\frac{1}{\pi} \arccos z \geq 0.8785 (1-z)/2$ FOR $z \in [-1,1]$

[BY MINIMIZING $\frac{2}{\pi} \arccos z / (1-z)$] $\square$

CONCLUSION:

$$\sum_{ij \in E} \frac{1}{\pi} \arccos u_i^T u_j \geq 0.8785 \sum \frac{1 - u_i^T u_j}{2}$$
$$\geq 0.8785 (OPT(P) - \varepsilon)$$
$$\geq 0.878\, OPT(P)$$

FOR $\varepsilon \leq 5 \cdot 10^{-4}$.

QUESTION: IS 0.878 A GOOD RESULT?

MAYBE YES.

- IF $P \neq NP$ THE BOUND 0.954 ...
- IF UNIQUE GAME CONJECTURE ... 0.878

PROBLEM MAX-2-LIN (MOD q)

INPUT: $x_1 \dots x_n \in \{0, 1 \dots q-1\}$

$$x_i - x_j = c_\ell$$

$$\text{EX.}\quad \left.\begin{array}{l} x_1 - x_2 \equiv 50 \pmod{97} \\ x_3 - x_5 = 3 \pmod{97} \end{array}\right\} m$$

TASK: FIND $x_1, x_2 \dots x_n$ SATISFYING MAX # OF EQUATIONS

(IN q^m TIME CLEAR)

- $(1-\delta, \varepsilon)$-GAP VERSION

$\geq (1-\delta) m$ SATISFIED ... YES

$\leq \varepsilon m$ SATISFIED ... NO

OTHER ... ANYTHING.

UGC:

$\forall \varepsilon > 0 \ \exists q$ S.T. $(1-\varepsilon, \varepsilon)$-GAP VERSION NOT IN POLYNOMIAL TIME UNLESS $P = NP$.

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