Math-589 Homework #2

I will finish the homework before 3pm Nov 26. If I spot a mistake I will let the lecturer know as soon as possible. I will type the solution.

1: (*Carathédory's theorem*)
Let $X \subset \mathbb{R}^d$ and let $\mathbf{x}$ be in the convex hull of X. Show that $\mathbf{x}$ is a convex combination of at most $d+1$ points of X.

2: (*Polytopes*)
Let $P \subset \mathbb{R}^4 = conv\{\pm \mathbf{e}_i \pm \mathbf{e}_j : i \neq j, \; i,j = 1,2,3,4\}$, where $\mathbf{e}_1, ..., \mathbf{e}_4$ is the standard basis. (This P is called the 24-cell.) Describe the face lattice of P. Describe its dual $P^* := \{\mathbf{y} \in \mathbb{R}^4 : \mathbf{x}^T\mathbf{y} \leq 1 \text{ for all } \mathbf{x} \in P\}$.

3: (*What is quaziconvex?*)
Let $f : \mathbb{R}^n \to \mathbb{R}$ be a differentiable function with convex domain. Show that f is quaziconvex if and only if for all $\mathbf{x}, \mathbf{y} \in dom(f)$

$$f(\mathbf{y}) \leq f(\mathbf{x}) \implies \nabla f(\mathbf{x})^T(\mathbf{y} - \mathbf{x}) \leq 0.$$

4: (*Generalized posynomials and geometric programming.*)
Show that generalized geometric program cannot express more than a geometric program. That is, for a program

$$(GGP) \begin{cases} \text{Minimize} & h_0(\mathbf{x}) \\ \text{subject to} & h_i(\mathbf{x}) \leq 1, i = 1, \ldots, m \\ & g_i(\mathbf{x}) = 1, i = 1, \ldots, p \\ & \mathbf{x} > \mathbf{0} \end{cases}$$

where h_i is a generalized posynomial and g_i is monomial find a geometric program (GP) that has the same solution but uses only posynomials and monomials.

5: (*KKT conditions*)
Solve the following program (P) using KKT conditions.

$$(P) \begin{cases} \text{Minimize} & f(x_1, x_2) = x_1^2 - 2x_1 + x_2^2 + 1 \\ \text{subject to} & g_1(x_1, x_2) = x_1 + x_2 \leq 0 \\ & g_2(x_1, x_2) = x_1^2 - 4 \leq 0 \\ & x_1, x_2 \in \mathbb{R} \end{cases}$$

6: (*Geometric program*)

Solve the following constrained geometric program (GP).

$$(GP)\quad \begin{cases} \text{Minimize} & x^{1/2}+y^{-2} \\ \text{subject to} & x^{-1}z+x^{-1}w \leq 1 \\ & yz^{-1}+wz^{-1} \leq 1 \\ & x>0, y>0, z>0, w>0 \end{cases}$$

7: (*What is coercive?*)
Decide for which $a \in \mathbb{R}$ the following function is coercive:

$$f(x,y) = x^2 + axy + y^2$$

Hint: Recall that f is coercive if $\lim_{\|x\|\to\infty} f(x) = +\infty$.

8: (*A bit more coercive thinking*)
Let a continuos function $f(x,y)$ on $\mathbb{R}^2$ be such that for each real number t, we have

$$\lim_{x\to+\infty} f(x,tx) = \lim_{x\to-\infty} f(x,tx) = \lim_{y\to+\infty} f(ty,y) = \lim_{y\to-\infty} f(ty,y) = +\infty.$$

Is it true that $f(x,y)$ is coercive?

9: (*Penalty methods*)

a) Use the penalty function method with the Courant-Beltrami penalty term to solve the problem (P).

$$(P)\quad \begin{cases} \text{Minimize} & f(x_1,x_2) = x_1+x_2 \\ \text{subject to} & x_1^2 - x_2 \leq 2 \end{cases}$$

b) Show that the objective function $F_k(\mathbf{x})$ corresponding to the Absolute value penalty term has no critical points off the parabola

$$x_1^2 - x_2 = 2$$

for $k > 1$ and compute the minimizer of $F_k(\mathbf{x})$.