MATH 589
"NONLINEAR OPTIMIZATION"

NOT AN EXPERT
- AZA 484 ... MAYBE MORE PROOFS
  (NOT FIXED SYLLABUS)

- 6x HW NO EXAMS

- SHOULD ATTEND
- ASK, COMMENTS WHAT TO HEAR???
  THEORETICAL

- BOOKS
  NONLINEAR OPTIMIZATION
  A. RUSZCZYŃSKI
  CONVEX OPTIMIZATION
  S. BOYD AND L. VANDENBERGHE
  (ONLINE PDF)

LECTURES ON DISCRETE GEOMETRY
J. MATOUŠEK

OPTIMIZATION .... #1 INVENTION OF 1900-2000

$$
(P)\quad \begin{cases} \text{MINIMIZE } f(x) & x \in \mathbb{R}^n \\ \text{S.T. } g_1(x) \le b_1 & \\ \quad\;\; g_2(x) \le b_2 & g_i : \mathbb{R}^n \to \mathbb{R} \\ \quad\;\; \vdots & b_i \in \mathbb{R} \\ \quad\;\; g_m(x) \le b_m & \end{cases}
$$

GENERAL, SUPER DIFFICULT

f, g LINEAR   LINEAR PROGRAM
→ SOLVABLE EFFICIENTLY

INVESTIGATE WHEN SOLVABLE

WHO KNOWS LP ???

IN THEORY YOU MIGH USE PROGRAMS
LATER

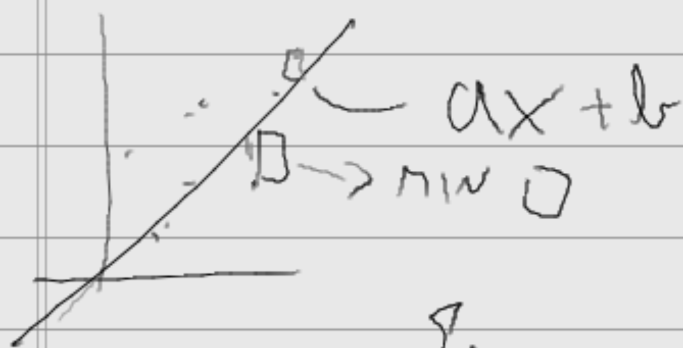
- EXAMPLES:
  DIET PROBLEM
  FOOD $x_1, x_2 \ldots x_n$ $\{1 \ldots m\}$
  x_i CONTAINS a_{ji} OF NUTRITION $j \in$
  x_i COSTS c_i
  WANT OF NUTRITION j ATLEAST b_j

$$
(LP) \begin{cases} \text{MIN } \sum c_i x_i & \ldots \text{ COST} \\ \text{S.T. } \sum_i a_{ji} x_i \geq b_j & \forall j \\ x_i \geq 0 & \forall i \end{cases}
$$

FOR US ARMY

AS HW LATER

- FITTING A CURVE LEAST SQUARES
  POINTS $\{x_{i_1}, x_{i_2}\}$ FIND
  LINE THROUGH THEM



$$\text{MIN} \sum (a x_{i_1} + b - x_{i_2})^2$$

UNCONSTRAINED OPTIMIZATION

- FIND MIN & MAX OF
  $f(x) = 3x^4 - 2x^3 - 2x + 1$
  $x \in \mathbb{R}$

→ USE $f'(x) = 0$ & $f''(x) > 0$
$f''(x) < 0$

→ MINIMIZER & MAXIMIZERS

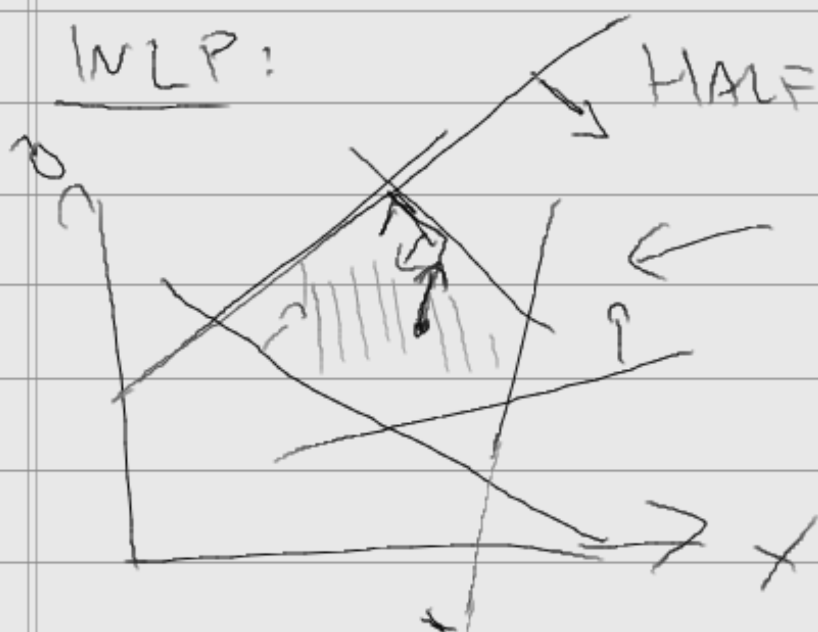
CALCULUS - UNCONSTRAINED NONLINEAR OPTIMIZATION

– GOOD IF f' AND f'' EXIST

– IF f CONVEX ... AT MOST 1 MAX/MIN

– HOW DOES IT LOOK IF f CONVEX →
→ DIRECTION OF DESCEND

INLP:



LINEAR ⇒ OPTIMUM IS IN VERTEX
→ STUDY OF CONVEX SETS & FUNCTIONS

NOTIONS

$\mathbb{R}^d$ .. d-DIMENSIONAL SPACE OF REALS

d-TOUPLES

HYPERPLANE

.. $d-1$ -DIMENSIONAL SUBSPACE

$\{x \in \mathbb{R}^d : \langle a | x \rangle = b,\ b \in \mathbb{R},\ a \in \mathbb{R}^d \setminus \{0\}\}$

DOT PRODUCT $= a_1 x_1 + a_2 x_2 + \dots a_d x_d$

[LINE IN 2D]

CLOSED HALF SPACE

$\{x \in \mathbb{R}^d : \langle a | x \rangle \leq b,\ b \in \mathbb{R},\ a \in \mathbb{R}^d \setminus \{0\}\}$

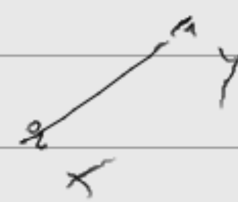
--> IN LP

CONVEXITY

$C \subseteq \mathbb{R}^d$ C IS CONVEX IF

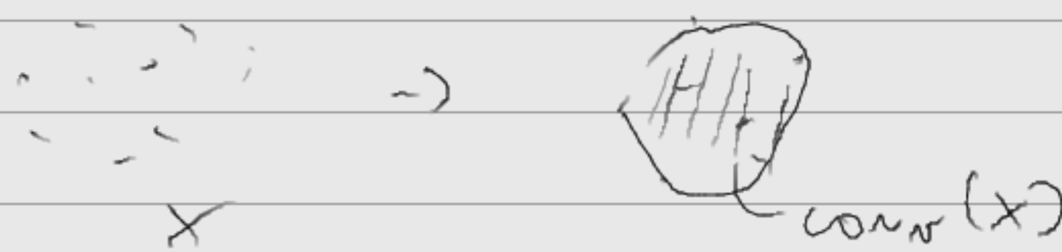
$\forall x, y \in C \Rightarrow$ (SEGMENT xy) $\in C$

$tx + (1-t)y \in C \quad (\forall t \in [0, 1])$



INTERSECTION OF FAMILY OF CONVEX SETS IS A CONVEX SET.
(PF RIGHT FROM DEFINITION)

DEF:
CONVEX HULL OF SET X = $\mathrm{conv}(X)$ IS INTERSECTION OF ALL CONVEX SETS CONTAINING X.



CLAIM

$x \in \mathrm{conv}(X) \iff \exists n\ \exists x_1 \dots x_n \in X$
$\exists t_1 \dots t_n \in \mathbb{R}\ \ t_i \geq 0\ \ \sum t_i = 1$
S.T. $x = \sum_{i=1}^{n} x_i t_i$ ← CONVEX COMBINATION OF $x_1 \dots x_n$

PROOF

$$Y = \left\{ \sum_{i=1}^{n} t_i x_i \quad n \in \mathbb{N},\ t_1 \ldots t_n \in \mathbb{R},\ t_i \geq 1 \quad \sum t_i = 1 \right\}$$

- $Y \subseteq \mathrm{conv}(X)$

CLEAR FOR $n = 1, 2$

MORE BY INDUCTION ON n

$y \in Y$

$\Leftarrow$ DO "PROJECTION"

$n \to n+1$

$$y = \sum_{i=1}^{n} t_i x_i \qquad \text{WLOG } 0 < t_1 < 1$$

$$= \underbrace{t_1 x_1 + (1-t_1) \underbrace{\sum_{i=2}^{n} \frac{t_i}{(1-t_1)} x_i}_{\in \mathrm{conv}(X) \text{ BY IND.}}}_{\text{CONV COMB OF 2 POINTS}}$$

$\mathrm{conv}(X) \subseteq Y$

- $X \subseteq Y$ ✓
- Y convex ?

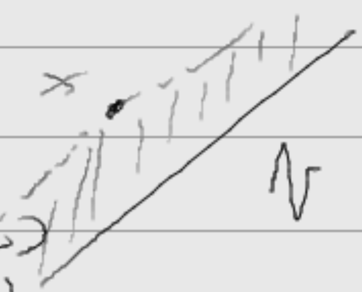
$$\left.\begin{aligned} x &= \sum t_i x_i \\ x' &= \sum_{i=1} t'_i x_i \end{aligned}\right\} \quad tx + (1-t)x' =$$

$$= \sum \underbrace{\left(t\,t_i + (1-t)t'_i\right)} x_i$$

$$t \underbrace{\sum t_i}_{=1} + (1-t)\underbrace{\sum t'_i}_{=1} = 1$$

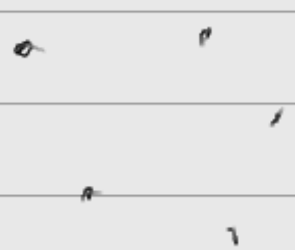
ex $hv\{e\}$

LINE ⇗ ↑ POINT



$\mathrm{conv}(hv\{e\})$

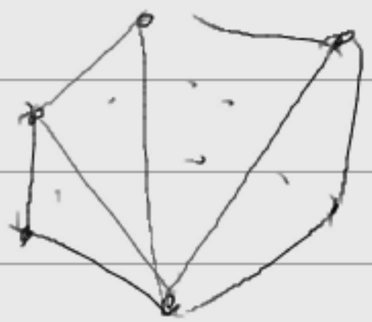
X



$\mathrm{conv}(X)$

THM CARATHÉODORY

$X \subseteq \mathbb{R}^d$, $x \in \mathrm{conv}(X)$, THEN x IS A CONVEX COMBINATION OF AT MOST $d+1$ POINTS OF X



PROOF BY INDUCTION