$\mathrm{conv}(X) \subseteq Y$

- $X \subseteq Y$ ✓
- Y convex ?

$$\left.\begin{aligned} x &= \sum t_i x_i \\ x' &= \sum_{i=1}^{n} t'_i x_i \end{aligned}\right\} \quad t x + (1-t) x' =$$

$$= \sum \underbrace{\left(t t_i + (1-t) t'_i\right)}_{} x_i$$

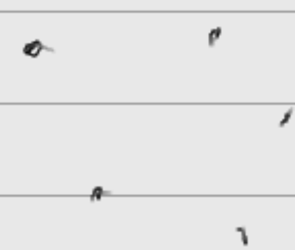
$$t \underbrace{\sum t_i}_{=1} + (1-t) \underbrace{\sum t'_i}_{=1} = 1$$

Ex $\ell \cup \{e\}$

LINE POINT

$\mathrm{conv}(\ell \cup \{e\})$

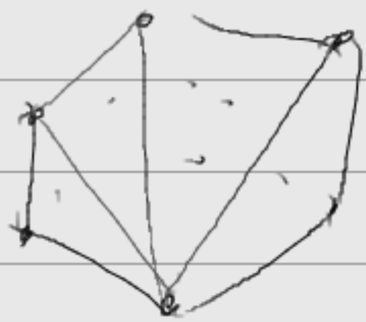
X



$\mathrm{conv}(X)$

THM CARATHÉODORY

$X \subseteq \mathbb{R}^d$, $x \in \text{conv}(X)$, THEN x IS A CONVEX COMBINATION OF AT MOST $d+1$ POINTS OF X



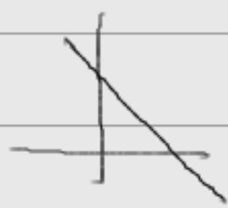
PROOF BY INDUCTION

AFFINE SUBSPACES

- LINEAR SUBSPACE

2D

LINES NOT THROUGH THE ORIGIN ??



← AFFINE SUBSPACE

AFFINE SUBSPACE IS LINEAR SUBSPACE + N

AFFINE HULL OF $X \subseteq \mathbb{R}^d$

INTERSECTION OF ALL AFFINE SUBSPACES CONTAINING X (SMALLEST ONE) $\mathrm{aff}(X)$

AFFINE COMBINATION OF $x_1 \dots x_n \in \mathbb{R}^d$ WITH $t_1 \dots t_n$, $\sum_{i=1}^{n} t_i = 1$ IS

$$\sum_{i=1}^{n} t_i x_i \in \mathbb{R}^d$$

$\boxed{\mathrm{aff}(X) = \text{ALL AFF COMB}}$

VECTORS $x_1 \dots x_n$ ARE AFFINE DEPENDENT IF $\exists i$ S.T. x_i IS AFF. COMBINATION OF THE REST

$x_1 \ldots x_n$ AFF DEP

$$x_1 = \sum_{i=2}^{n} t_i x_i$$

$$0 = -x_1 + \sum_{i=2}^{n} t_i x_i \qquad 1 = \sum_{i=2}^{n} t_i$$

$$0 = \sum_{i=2}^{n} t_i (x_i - x_1)$$

IFF $(x_i - x_1)$ ARE LIN. DEPENDENT

DEF IF

$$0 = \sum_{i=1}^{n} t_i x_i \quad \& \quad \sum t_i = 0$$

LIN DEP IS WITHOUT $\sum t_i = 0$

$\Rightarrow$ AT MOST $d+1$ AFF. INDEP POINTS IN $\mathbb{R}^d$

AFF HULL OF k AFF INDEP POINTS HAS DIMENSION $k-1$

THEOREM RADON

LET A BE A SET OF $d+2$ POINTS IN $\mathbb{R}^d$

THEN $\exists A_1, A_2 \subseteq A$, $A_1 \cap A_2 = \emptyset$

SUCH THAT $\operatorname{conv}(A_1) \cap \operatorname{conv}(A_2) \neq \emptyset$

EX

$d = 2$

PROOF:

$A = \{x_1, x_2 \ldots x_{d+2}\} \Rightarrow$ AFF DEPENDENT.

$\exists (t_1, \ldots t_{d+2}) \neq (0, 0 \ldots 0)$, $\sum t_i = 0$

& $\sum t_i x_i = 0$

LET $A_1 = \{x_i \in A : t_i > 0\}$

$A_2 = \{x_i \in A : t_i < 0\}$

$$\sum_{x_i \in A_1} t_i x_i = -\sum_{x_i \in A_2} t_i x_i \qquad \sum t_i = 1 \text{ FOR CONV. COMB}$$

$$s = \sum_{x_i \in A_1} t_i \qquad \sum_{x_i \in A_1} \frac{t_i}{s} x_i = \sum_{x_i \in A_2} \frac{-t_i}{s} x_i \in \operatorname{conv}(A_1) \cap \operatorname{conv}(A_2)$$

$\square$

THEOREM HELLY - INTERSECTIONS OF CONVEX SETS

LET $C_1 \dots C_n$ BE CONVEX SETS IN $\mathbb{R}^d$, $n \geq d+1$. IF INTERSECTION OF EVERY $d+1$ OF THEM NONEMPTY THEN $\bigcap_{i=1}^{n} C_i \neq \emptyset$.

$n \geq d+1$ NEEDED

EX INTERSECTION OF d NOT ENOUGH 

PROOF: INDUCTION ON n

1. $n = d+1$ ✓
2. $n \geq d+2$ $\forall j \in (1 \dots n)$ $\bigcap_{i \neq j} C_i \neq \emptyset$

$\forall_j$: $a_j \in \bigcap_{i \neq j} C_i$

$a_j \notin C_j$ BUT IN ALL OTHERS

POINTS $a_1 \dots a_{d+2}$ RADON $\Rightarrow$

$\exists I_1, I_2 \subseteq \{1, \dots d+2\}$ DISJOINT S.T.

$x \in \mathrm{conv}\{a_j : j \in I_1\} \cap \mathrm{conv}\{a_j \; j \in I_2\}$

(WANT x IN ALL OF C_i)

I) $i \in \{1, 2 \dots n\} \setminus I_1$, $\forall_j \in I_1$, $a_j \in C_i$

$\Rightarrow x \in C_i$ AS $x \in \mathrm{conv}\{a_j\}$

II) $i \in \{1, 2 \dots n\} \setminus I_2$ $\forall_j \in I_2$ $a_j \in C_i$

$\Rightarrow x \in C_i$ AS $x \in \mathrm{conv}\{a_j\}$

$x \in \bigcap C_i$

$\square$

RESTATE:

CONVEX $C_1 \dots C_n$ & $\bigcap_{i=1}^{n} C_i = \emptyset \implies$

$\implies \exists$ AT MOST $d+1$ OF THEM WITNESSING

INFINITE VERSION

CONVEX $C_1 \dots\dots$ ALL $d+1$ NONEMPTY $\cap$

NO EVEN IN $\mathbb{R}^1$

$\{(0, 1/n) \quad n \in \mathbb{N}\}$ ... NOT CLOSED

$\{[n, \infty) \quad n \in \mathbb{N}\}$ ... NOT BOUNDED

THEOREM INFINITE HELLY

LET $\mathcal{C}$ BE A FAMILY OF COMPACT ($|\mathcal{C}| \geq d+1$)
CONVEX SETS IN $\mathbb{R}^d$ ST. EVERY $d+1$ HAVE
NONEMPTY INTERSECTION. THEN ALL
SETS IN $\mathcal{C}$ HAVE A NONEMPTY INTERSECTION.

PROOF

FINITE INTERSECTIONS EXIST

THEN COMPACTNESS

$\square$