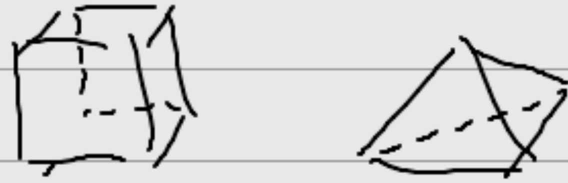
CONVEX POLYTOPES



DEF:

H-POLYHEDRON ... $\bigcap_{H \in \mathcal{H}} H$

$\mathcal{H}$: FINITE SET OF CLOSED HALFSPACES IN $\mathbb{R}^d$

(NOT BOUNDED MAYBE)

H-POLYTOPE .. BOUNDED H-POLYHEDRON

V-POLYTOPE conv(V)

$V \subseteq \mathbb{R}^d$, $|V|$ IS FINITE

EX CUBE - $\bigcap$ 6 HALFSPACES IN 3D

conv OF 8 POINTS

H-POLYHEDRON ... CONSTRAINS OF L.P.

$\min f(x)$

S.T. $2x_1 + 3x_2 \le 2$

$\vdots$

$x_1 \ge 0$

← ALL HALFSPACES

SOLUTION .. X - VERTEX OF H-POLYHEDRON

$H \to V$??

THEOREM

EACH V-POLYTOPE IS H-POLYTOPE

EACH H-POLYTOPE IS V-POLYTOPE

DEF

DIMENSION OF POLYTOPE/ POLYHEDRON P

IS DIM. OF $\mathrm{aff}(P)$ ... AFINE HULL

PROOF:

- H-POLYTOPE IS V-POLYTOPE

BY INDUCTION ON d ~ $\mathbb{R}^d$

$d=1$ ... CLEAR

UND $P=\mathrm{conv}(V)$

$d \geq 2$

HALFSPACES $\mathcal{H}$ (FINITE), $P=\bigcap \mathcal{H}$

$\forall \gamma \in \mathcal{H}$ LET $F_\gamma = P \cap \partial\gamma$

BOUNDARY

F_γ IS H-POLYTOP, $\dim(F_\gamma) \leq d-1$

LIVES IN $\partial\gamma$

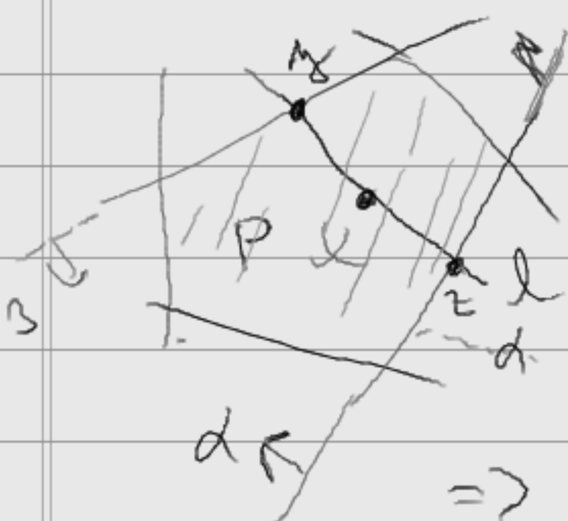
WHICH IS AFFINE

BY INDUCTION $\exists\, V_\gamma \subseteq F_\gamma$ FINITE

S.T. $F_\gamma = \mathrm{conv}(V_\gamma)$

CLAIM: $V = \bigcup V_\gamma$ AND $P = \mathrm{conv}(V)$

LET $x \in P$ [FIND $\in \mathrm{conv}(V)$]



ℓ .. ANY LINE

.. $z \in \partial\alpha \cap P$

$y \in \partial\beta \cap P$

$\Rightarrow z \in \mathrm{conv}(V_\alpha)$

$y \in \mathrm{conv}(V_\beta)$

$\Rightarrow x \in \mathrm{conv}(V_\alpha \cup V_\beta) \subseteq \mathrm{conv}(V)$

- EVERY V POLYTOPE IS H-POLYTOPE

AS $\boxed{\text{HW}}$

- FROM DUALITY
- FROM TAKING $\binom{V}{d}$ THAT DO NOT CUT P.

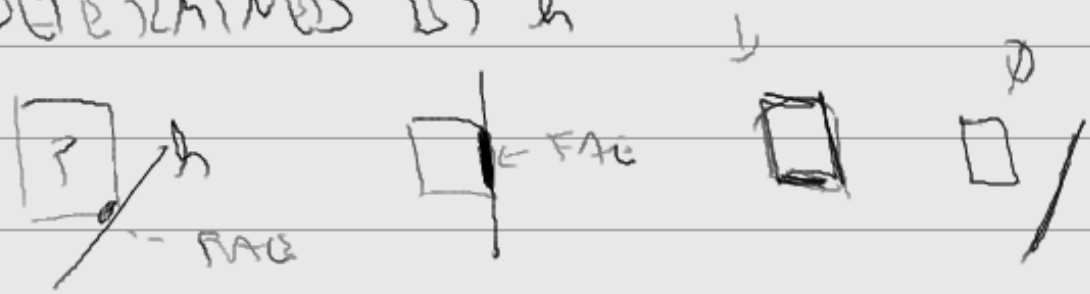
$\square$

DEF:

FACE OF A POLYTOPE P IS

1, P ITSELF

2, $P \cap h$ WHERE h IS A HYPERPLANE S.T. P IS IN ONE OF HALFSPACES DETERMINED BY h



* EVERY FACE IS A POLYTOP

AFFINE HULL

DEF

DIMENSION OF POLYTOPE P IS $\dim \operatorname{aff} P$

→ SAME ABOUT FACES

$\dim$ VERTICES $= 0$

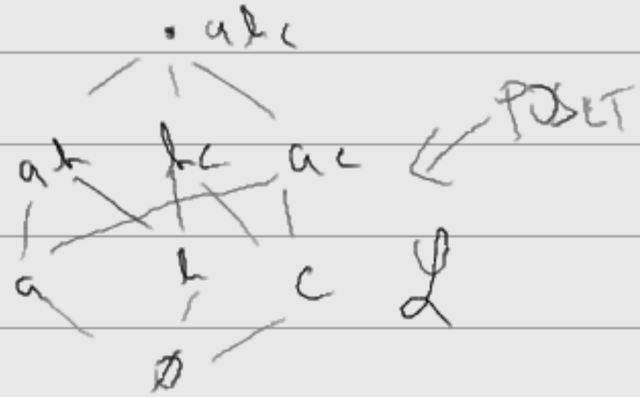
$\dim \emptyset = -1$

$\dim = 1$ ... EDGES

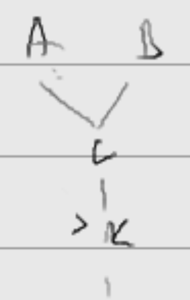
$\dim = d-1$ ... FACETS

NOTE FOR POSETS

FACES AS SUBSETS OF VERTICES, ORDER BY INCLUSION



- POLYTOPES P & L COMB. EQUIV IF SAME POSETS
- IS ACTUALLY A LATTICE



- MEET $\forall A,B \in \mathcal{L}\ \exists C \in \mathcal{L}$
  $C \leq A,\ C \leq B\ \&\ \forall K \in \mathcal{L}$
  $K \leq A\ \&\ K \leq B \Rightarrow K \leq C$
- JOIN $\forall A,B \in \mathcal{L}\ \exists C \in \mathcal{L}$ S.T.
  $C \geq B\ \&\ C \geq B\quad \forall K \in \mathcal{L}$
  $K \geq A\ \&\ K \geq B \Rightarrow K \geq C$

→ MEET OF 2 FACES IS THEIR INTERSECTION

- ALL MAX. CHAINS HAVE THE SAME LENGTH

• ATOMIC .. EVERY FACE IS JOIN OF ITS VERTICES

• CO ATOMIC .. EVERY FACE IS MEET OF ALL FACETS CONTAINING IT

⇒ ENOUGH TO KNOW FACETS & VERTICES TO RECONSTRUCT THE WHOLE POSET

• NO FULL CHARACTERIZATION KNOWN WHEN POSET CORRESPONDS TO POLYTOP

FOR POLYTOP P ITS DUAL P^* IS A POLYTOP WITH $\mathcal{L}$ FLIPPED UPSIDE DOWN. FACETS ⟺ VERTICES

– EXISTS PRECISE GEOMETRIC CONSTRUCTION – P° IS REALLY A POLYTOP

P CONTAINS ORIGIN. THEN

$$P^\circ = \{ y \in \mathbb{R}^n : \langle y | x \rangle \le 1 \quad \forall x \in P \}$$

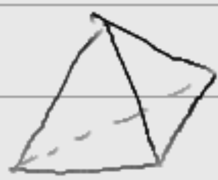
SEE PART $\ell = 1$

VERTICES → INTERSECTION OF HALFSPACES

NOTE FOR GRAPH THEORY

POLYTOP P KEEP VERTICES & EDGES =>

=> GRAPH



$\rightarrow K_4$

THEOREM [STEINITZ]

A FINITE GRAPH G IS ISOMORPHIC TO A GRAPH OF 3D POLYTOP IFF G IS PLANAR AND VERTEX 3-CONNECTED

- NO SIMPLE PROOF KNOWN
- NO CHARACTERIZATION KNOWN FOR HIGHER DIMENSION
- RECOGNITION OF 4D ONES CONJECTURED TO BE NP-C

$\rightarrow$ SOLUTION => INSTANT A+ FROM 584

[KNOWN MUST BE d-CONNECTED] POLYT P

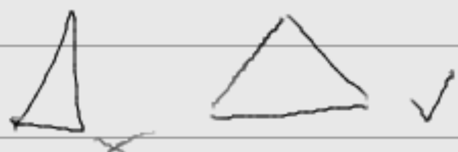
- DUAL OF 3-CONNECTED PLANAR GRAPH IS GRAPH OF P^*

DEF:

SIMPLEX .. CONVEX HULL OF AFF. INDEPENDENT POINTS.

$d = 0$ 1   2   3   ← DIM

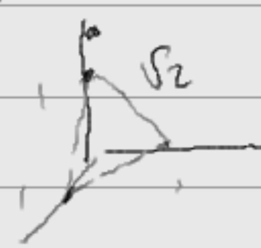
REGULAR SIMPLEX .. EQUAL DISTANCES



REGULAR d-DIMENSIONAL SIMPLEX:

$\mathrm{conv}\{e_1, e_2, \dots, e_{d+1}\} \subseteq \mathbb{R}^{d+1}$

$e_i = (0, 0, \dots, \underset{\uparrow\, i}{1}, 0, 0, \dots, 0)$



DEF

NOT P & $\emptyset$

POLYTOPE P IS SIMPLICIAL IF EVERY PROPER FACE IS SIMPLEX

(ENOUGH IF ALL FACETS ARE SIMPLICES)

EX NOT CUBE

YES CROSSPOLYTOPE

$\Rightarrow \mathrm{conv}\{\pm e_1, \pm e_2, \ldots, \pm e_d\}$ 

VERTICES OF P IN GENERAL POSITION

$\Rightarrow$ P IS SIMPLICIAL

GEN. POSITION: NO j POINTS IN $(j-2)$-DIM AFF. SUBSPACE FOR $j = 3 \ldots d+1$

EX $d=2$: GENERAL POSITION .. NO 3 POINTS ON A LINE.

$\to$ FACET IS CONV OF ITS VERTICES. NOT SIMPLEX $\Rightarrow$

$\Rightarrow \geq d+1$ VERTICES IN $d-1$ DIM. AFF-SUBSPACE.

DEF POLYTOPE P IS SIMPLE IF EVERY j-DIM. FACE IS IN $d-j$ FACETS.

CLAIM

P SIMPLICIAL $\Leftrightarrow$ P^* SIMPLE

CROSSPOLYTOPE $\Leftrightarrow$ HYPERCUBE*

d-CUBE*

~ CONNECTION TO LP ~

MAX $c^T x$ 1 ROW = 1 HALFSPACE

S.T. $Ax \le b$

$c \in \mathbb{R}^d$, $A \in \mathbb{R}^{m \times d}$, $b \in \mathbb{R}^m$

$x \in \mathbb{R}^d$

-- $Ax \le b$ IS H-POLYTOPE P

.. OPTIMUM IN VERTEX OF P

SIZE OF PROGRAM .. |A| .. # ROWS ..

- HOW MANY CANDIDATES FOR x ? VERTICES?

HYPERCUBE d-DIM. $\cap$ OF 2d - HALFSP.

BUT HAS 2^d VERTICES ! -C

n FACETS ... $\Theta(n^{\lfloor \frac{d}{2} \rfloor})$ IS MAX # OF VERTICES

WE - LOWER BOUND IN DUAL SETTING. (n ≈ d BAD)

$\Rightarrow$ SOLVING LP NOT EASY

- SIMPLEX METHOD FOR SOLVING LP.

FIND VERTEX OF $Ax \le b$. TRAVERSE

EDGES „TOWARDS" OPTIMUM.

- IS THIS FAST IF EDGES PICKED „SMARTLY" ?

OR GIVES LOWER BOUND

CONJECTURE HIRSCH

GRAPH OF POLYTOPE OF DIM. d WITH

n FACETS HAS DIAMETER $n-d$.
[- WOULD GIVE LINEAR # OF STEPS]
IMPLIED BY $n = 2d$ .. d-STEP CONJECTURE
- ALSO FOR A+ FROM CLASS :-)
[ENOUGH BOUND POLYNOM IN n & d]
BEST BOUND IS $2n^{\log d+1}$

CYCLIC POLYTOPES

MOMENT CURVE