~CONNECTION TO LP~

$$\max c^T x$$

$$\text{S.T. } Ax \leq b$$ ← 1 ROW = 1 HALFSPACE

$c \in \mathbb{R}^d$, $A \in \mathbb{R}^{m \times d}$, $b \in \mathbb{R}^m$

$x \in \mathbb{R}^d$

-- $Ax \leq b$ IS H-POLYTOPE P

-- OPTIMUM IN VERTEX OF P

SIZE OF PROGRAM -- $|A|$ -- # ROWS --

- HOW MANY CANDIDATES FOR x? VERTICES?
  HYPERCUBE d-DIM. $\cap$ OF $2d$ - HALFSP.
  BUT HAS 2^d VERTICES !-(
  n FACETS -- $\Theta(n^{\lfloor d/2 \rfloor})$ IS MAX # OF VERTICES
  WE- LOWERBOUND IN DUAL SETTING. ($n \approx d$ BAD)

- SIMPLEX METHOD FOR SOLVING LP. | ⇒ SOLVING LP NOT EASY
  FIND VERTEX OF $Ax \leq b$. TRAVERSE
  EDGES „TOWARDS" OPTIMUM.
  - IS THIS FAST IF EDGES PICKED „SMARTLY"? OR GIVES LOWER BOUND

CONJECTURE HIRSCH

GRAPH OF POLYTOPE OF DIM. d WITH

n FACETS HAS DIAMETER $n-d$.

[- WOULD GIVE LINEAR # OF STEPS]

IMPLIED BY $n = 2d$ .. d-STEP CONJECTURE

- ALSO FOR A+ FROM CLASS :-)

[ENOUGH BOUND POLYNOM IN n & d]

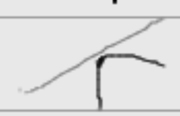
BEST BOUND IS $2n^{\log d+1}$

<u>CYCLIC POLYTOPES</u> - FEW VERTICES, MANY FACETS (DUAL)

MOMENT CURVE

$$\gamma = \{(t, t^2, t^3, \dots t^d) : t \in \mathbb{R}\}$$

LEMMA:

HYPERPLANE H AND MOMENT CURVE γ INTERSECT IN $\leq d$ POINTS. IF IN $= d$ POINTS THEN NONE OF THEM IS TANGENT  NO

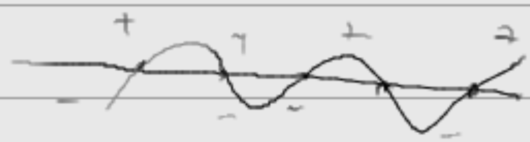
PROOF

H .. $\langle a | x \rangle = b$ INTERSECTION WITH γ.

$$a_1 t + a_2 t^2 + \dots a_d t^d - b = 0$$

↳ POLYNOM OF DEGREE d $\Rightarrow$ AT MOST d

ROOTS. IF ALL d ARE DISTINCT, THEN IT MUST CHANGE SIGN ALWAYS.



$\square$

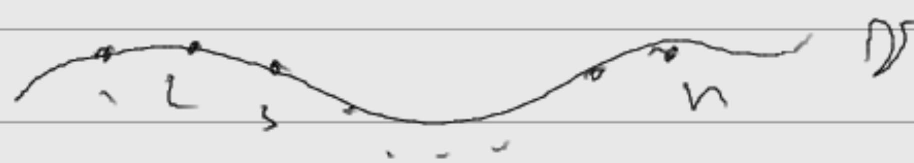
$\Rightarrow$ EVERY d POINTS ON MOMENT CURVE ARE AFF. INDEPENDENT (OTHERWISE WE COULD ADD 1 MORE POINT TO INTERS. WITH A HYPERPLANE)

- EXAMPLE OF SET OF POINTS IN „GENERAL POSITION"

DEFINITION

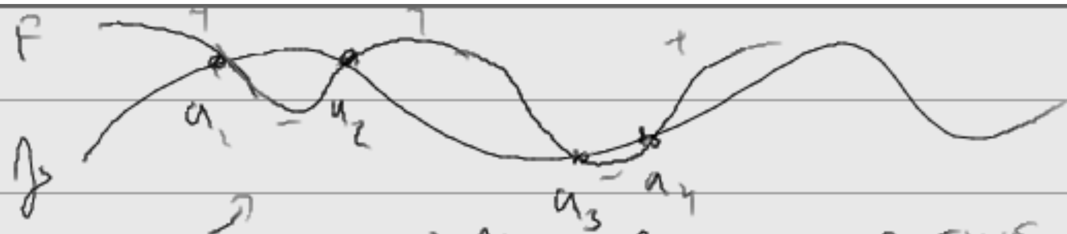
$V \subseteq \gamma$, $|V|$ FINITE, $\mathrm{conv}(V)$ IS CYCLIC POLYTOPE

LOOKING FOR FACETS:



FACET .. $a_1\, a_2\, a_3 \,\ldots\, a_d = F$

HOW THEY LOOK?



ALWAYS TAKE 2 CONSECUTIVE POINTS

[MAYBE BEGINNING & END EXCEPTION]

POLYTOPE MUST BE ON ONE SIDE OF F

ROUGHLY. PICK $\frac{d}{2}$ PAIRS FROM n POINTS

$$\simeq \binom{n-\frac{d}{2}}{\frac{d}{2}} \simeq n^{\frac{d}{2}}$$

↳ POSSIBLE TO COUNT MORE PRECISELY

– EX FROM COMBINATORICS.

THEOREM

CYCLIC POLYTOPE MAXIMIZES # OF FACES IN EVERY DIMENSION (AMONG POLYTOPES WITH n VERTICES – FIXED)

THEOREM SEPARATION THEOREM

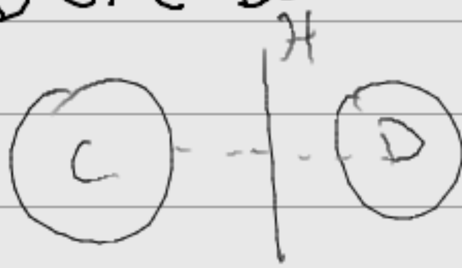
LET $C, D \subseteq \mathbb{R}^d$ CONVEX & $C \cap D = \emptyset$

THEN EXISTS HYPERPLANE H SEPARATING C AND D. $\exists\, a \in \mathbb{R}^d$ $b \in \mathbb{R}$ ST.

$\forall x \in C \quad \langle a | x \rangle \le b$

$\forall x \in D \quad \langle a | x \rangle \ge b$

STRICT SEPARATION IF C & D CLOSED AND ONE BOUNDED



PROOF: (SKETCH)

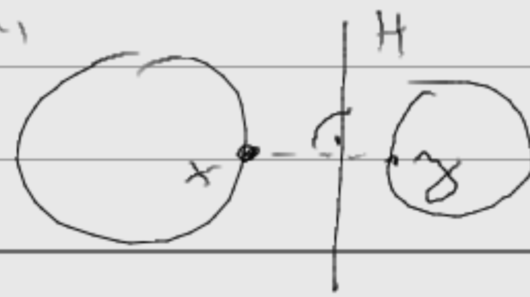
a) C, D COMPACT

DEF $C \times D$ $\quad f: C \times D \to \mathbb{R}$

$\|x - y\| \quad x \in C, y \in D$

COMPACT SET $\Rightarrow$ f HAS MINIMUM $\Rightarrow$

x, y ARE "CLOSEST"



b) C COMPACT, D CLOSED

$D' = D \cap B(0, M)$ FOR HUGE M)

$\Rightarrow y \in D'$ CLOSEST ALSO FOR WHOLE D IF M LARGE ENOUGH

$\rightarrow$ AGAIN STRICT SEPARATION

c) C, D ARBITRARY CONVEX

SUPPOSE $0 \in C$. DEFINE

$C_1 \subseteq C_2 \subseteq \dots \subseteq C$ ST.

$C_i = \text{CLOSURE OF}\left((1-\frac{1}{n})\, C \cap B(0,n)\right)$

MAKE D_i ANALOGOUSLY

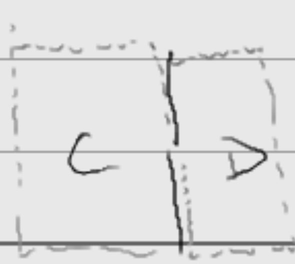
$\bigcup C_i = C$ & $\bigcup D_i = D$

C_i, D_i COMPACT. USE a, GET SERIES OF HYPERPLANES H_i SEPARATING C_i & D_i

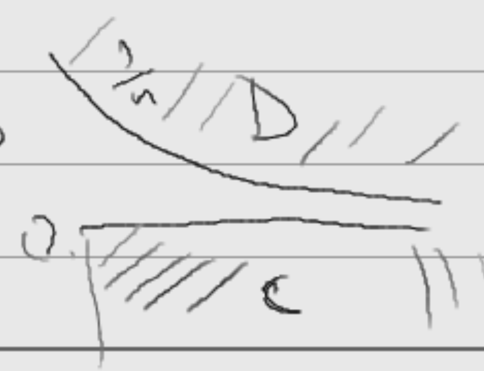
CAN BE SHOWN THAT H_i CONVERGES.

$\rightarrow$ NOT STRICT SEPARATION $\square$

NOTE CLOSED NEEDED



CANNOT BE STRICTLY SEPARATED $\rightarrow$



DUALITY FOR LINEAR PROGRAMMING

SUPPOSE

$$(P)\begin{cases} \text{MAX } 2x_1 + 3x_2 & \\ \text{S.T. } 4x_1 + 8x_2 \le 12 & (a) \\ 2x_1 + x_2 \le 3 & (b) \\ 3x_1 + 2x_2 \le 4 & \\ x_1, x_2 \ge 0 & \end{cases}$$

GIVE ME SOME UPPER BOUND ON SOLUTION

$$2x_1 + 3x_2 \le 4x_1 + 8x_2 \le 12$$

$$\le 2x_1 + 4x_2 \le 6 \quad \text{BETTER}$$

$$\le ((a)+(b)) \cdot \tfrac{1}{3} =$$

$$= \tfrac{1}{3}(6x_1 + 9x_2) = 2x_1 + 3x_2 \le \tfrac{1}{3}(3+12) = 5$$

WHAT IS THE BEST THAT YOU CAN DO ??

- BOUND $d_1x_1 + d_2x_2 \le h$

$d_1 \ge 2,\ d_2 \ge 3$ MINIMIZE h

$\to 2x_1 + 3x_2 \le d_1x_1 + d_2x_2 \le h$