DUALITY FOR LINEAR PROGRAMMING

SUPPOSE

$$
(P)\begin{cases}
\text{MAX } 2x_1 + 3x_2 & \\
\text{S.T. } 4x_1 + 8x_2 \le 12 & (a) \\
2x_1 + x_2 \le 3 & (b) \\
3x_1 + 2x_2 \le 4 & \\
x_1, x_2 \ge 0 &
\end{cases}
$$

GIVE ME SOME UPPER BOUND ON SOLUTION

$$2x_1 + 3x_2 \le 4x_1 + 8x_2 \le 12$$

$$\le 2x_1 + 4x_2 \le 6 \quad \text{BETTER}$$

$$\le \left((a)+(b)\right)\cdot\tfrac{1}{3} =$$

$$= \tfrac{1}{3}(6x_1 + 9x_2) = 2x_1 + 3x_2 \le \tfrac{1}{3}(3+12) = 5$$

WHAT IS THE BEST THAT YOU CAN DO ??

- BOUND $d_1x_1 + d_2x_2 \le h$

$$d_1 \ge 2, \; d_2 \ge 3 \quad \text{MINIMIZE } h$$

$$\to 2x_1 + 3x_2 \le d_1x_1 + d_2x_2 \le h$$

COMBINE INEQUALITIES: d_1 d_2

$$(4y_1 + 2y_2 + 3y_3)x_1 + (8y_1 + y_2 + 2y_3)x_2 \le$$

$$12y_1 + 3y_2 + 4y_3 = h \qquad y_i \ge 0 \text{ TO PRESERVE } \le$$

(D)
$$\begin{aligned} &\text{MINIMIZE } 12y_1 + 3y_2 + 4y_2 \\ &\text{S.T.} \quad 4y_1 + 2y_2 + 3y_3 \geq 2 \\ &\qquad\quad 8y_1 + y_2 + 2y_3 \geq 3 \\ &\qquad\quad y_1, y_2, y_3 \geq 0 \end{aligned}$$

PREVENT INEQUALITY SHIFT

(D) GIVES UPPER BOUNDS ON (P)

HOW GOOD ?? - EXACT :-)

$y = (\frac{5}{16}, 0, \frac{1}{4})$ ... 4.75

$x = (\frac{1}{2}, \frac{5}{4})$

DUALITY THM SAYS ALWAYS HOLDS

$$\begin{aligned} &\text{MAX } c^T x \\ &\text{S.T. } Ax \leq b \\ &\qquad x \geq 0 \end{aligned} \qquad\qquad \begin{aligned} &\text{MIN } b^T y \\ &A^T y \geq c \\ &y \geq 0 \end{aligned}$$

DUAL EXAMPLE

$$\begin{aligned} &\text{MIN } c^T x \\ &\text{S.T. } Ax = b \\ &\qquad x \geq 0 \end{aligned}$$

FIND DUAL

$$\min\ c^T x \qquad\qquad \max\ (b, -b)^T \binom{\mu}{\nu}$$
$$Ax \geq b \qquad\qquad (A^T, -A^T)\binom{\mu}{\nu} \leq c$$
$$-Ax \geq -b \quad \rightarrow \text{DUAL}$$
$$x \geq 0 \qquad\qquad \mu \geq 0,\ \nu \geq 0$$

SET $y = \mu - \nu \rightarrow \max\ b^T y$

$$A^T y \leq c$$

$\rightarrow y$ MAY BE NEGATIVE

DUALIZATION FOR EVERYONE

$A \in \mathbb{R}^{m \times n} \quad c \in \mathbb{R}^n \quad b \in \mathbb{R}^m$

	PRIMAL	DUAL
VARS	$x_1 \ldots x_n$	$y_1 \ldots y_m$
MATRIX	A	A^T
RIGHT-HAND	b	c
OBJECTIVE	$\max\ c^T x$	$\min\ b^T y$
CONSTRAINS	iTH CONSTR. $\leq$	$y_i \geq 0$
	HAS $\geq$	$y_i \leq 0$
	$=$	$y_i \in \mathbb{R}$

	$x_j \geq 0$	iTH CONSTRAINT $\geq$
	$x_j \leq 0$	$\leq$
	$x_j \in \mathbb{R}$	$=$

EX DUALITY FOR DIET PROBLEM

FOOD x_i OF COST c_i CONTAINS a_{ji} OF NUTRITIENT j AND WE WANT AT LEAST b_j OF NUTRITIENT j

$$(P) \begin{cases} \min & c^T x \\ \text{s.t.} & Ax \geq b \\ & x \geq 0 \end{cases}$$

$$(D) \begin{cases} \max & b^T y \\ & A^T y \leq c \\ & y \geq 0 \end{cases}$$

WHAT IS y???

y IS "PRICE OF NUTRITIENTS"

SUPPOSE YOU WANT TO MAKE

PILLS AND

- SELL THEM AS EXPENSIVE AS POSSIBLE
- PILLS ARE NOT MORE EXPENSIVE THAN REGULAR FOOD.

- FINDS BEST ASSIGMENT OF PRICES S.T. IT IS STILL BETTER TO USE PILLS

DUALITY THEOREM

FARKAS LEMMA

LET $A \in \mathbb{R}^{m \times n}$ AND $b \in \mathbb{R}^m$ EXACTLY ONE OF THE FOLLOWING HOLDS

a) — $\exists x \in \mathbb{R}^n$ $Ax = b, \; x \geq 0$ ← COMPONENTS

b) — $\exists y \in \mathbb{R}^m$ $y^T A \geq 0^T$ & $y^T b < 0$

↳ CLEARLY NOT BOTH AT THE SAME TIME

$$\underbrace{(y^T A)}_{\geq 0} \underbrace{x}_{\geq 0} = \underbrace{y^T b}_{< 0}$$

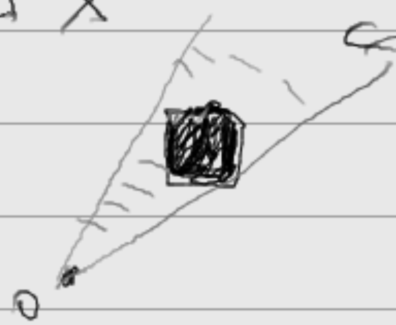
NOT POSSIBLE

DEF: $a_1, \ldots, a_n \in \mathbb{R}^m$ CONVEX CONE IS

$$\{t_1 a_1 + t_2 a_2 + \ldots + t_n a_n,\ t_i \geq 0\}$$

↳ LINEAR COMBINATIONS WITH NONNEGATIVE COEFICIENTS

- CONVEX CONE CAN BE DEFINED FOR ANY SET X



← CONE FOR ■

IF X FINITE CONVEX CONE CLOSED

NOT TRUE IF X INFINITE

OPEN HALFSPACE + ONE POINT

GEOMETRIC INTERPRETATION OF F.L.

$a_1, a_2, \ldots, a_n, b \in \mathbb{R}^m$. EXACTLY ONE OF

a) b IS IN CONVEX CONE GENERATED BY a_i

b, THERE EXISTS HYPERPLANE h
GOING THROUGH 0

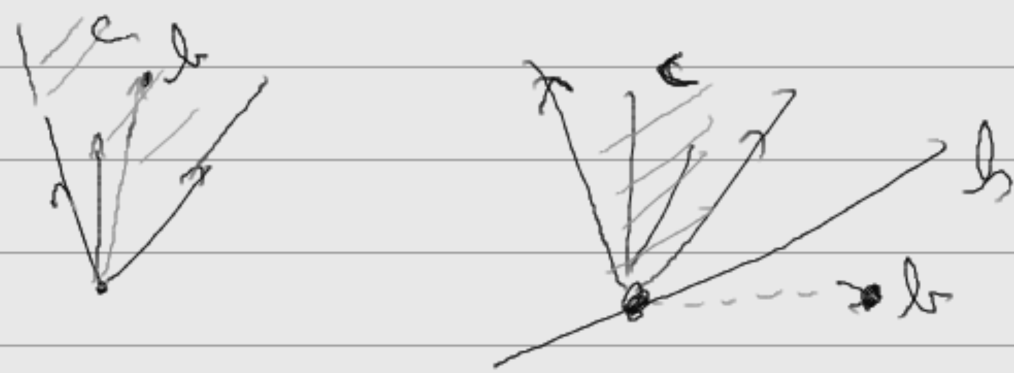
$$h = \langle y | x \rangle = 0$$

y ← FIXED

ST. $\forall i \; \langle y | a_i \rangle \geq 0$ .. CONE C IS
IN ONE HALF SPACE

AND $\langle y | b \rangle < 0$

" h SEPARATES CONE C AND b "



FROM SEPARATION THEOREM $\exists$
h': $\langle y | x \rangle \geq z \quad \forall x \in C$
& $\langle y | b \rangle < z$

NOTE $0 \in C$ .. $\langle y | 0 \rangle = 0 \geq z$

? CAN THERE BE $x \in C$ ST.
$\langle y | x \rangle < 0$?? IF YES

THEN $\langle y | 1000 x \rangle = 1000 \langle y | x \rangle < z$

$\Rightarrow$ SETTING $z = 0$ STILL SEPARATES :-)

REFORMULATIONS OF FARKAS LEMMA

a) $Ax = b$ HAS A NONNEGATIVE SOLUTION IFF $\forall y \in \mathbb{R}^m$ WITH $y^T A \geq 0^T$ ALSO $y^T b \geq 0$

b) $Ax \leq b$ HAS A NONNEGATIVE SOLUTION IFF $\forall y \in \mathbb{R}^m$ $y \geq 0$, WITH $y^T A \geq 0^T$ ALSO $y^T b \geq 0$

PROOF HW

$$(P) \begin{cases} \max \ c^T x \\ \text{s.t. } Ax \leq b \\ \phantom{\text{s.t. }} x \geq 0 \end{cases} \qquad (D) \begin{cases} \min \ b^T y \\ \text{s.t. } A^T y \geq c \\ \phantom{\text{s.t. }} y \geq 0 \end{cases}$$

LEMMA (WEAK DUALITY)

LET x, y BE FEASIBLE SOLUTIONS OF (P) AND (D) $\rightarrow$ SATISFYING S.T.

THEN $c^T x \leq b^T y$

$$c^T x = x^T c \leq x^T A^T y = (Ax)^T y \leq b^T y$$

$\square$