THEN $\langle y | 1000 x \rangle = 1000 \langle y | x \rangle < z$

$\Rightarrow$ SETTING $z = 0$ STILL SEPARATES :-)

REFORMULATIONS OF FARKAS LEMMA

a) $Ax = b$ HAS A NONNEGATIVE SOLUTION IFF $\forall y \in \mathbb{R}^m$ WITH $y^T A \geq 0^T$ ALSO $y^T b \geq 0$

b) $Ax \leq b$ HAS A NONNEGATIVE SOLUTION IFF $\forall y \in \mathbb{R}^m$ $y \geq 0$, WITH $y^T A \geq 0^T$ ALSO $y^T b \geq 0$

PROOF HW

$$(P) \begin{cases} \max & c^T x \\ \text{s.t.} & Ax \leq b \\ & x \geq 0 \end{cases} \qquad (D) \begin{cases} \min & b^T y \\ \text{s.t.} & A^T y \geq c \\ & y \geq 0 \end{cases}$$

LEMMA (WEAK DUALITY)

LET $x, y$ BE FEASIBLE SOLUTIONS OF (P) AND (D) $\hookrightarrow$ SATISFYING S.T.

THEN $c^T x \leq b^T y$

$$c^T x = x^T c \leq x^T A^T y = (Ax)^T y \leq b^T y$$

$\beta$

# THEOREM DUALITY OF LP

a) IF (P) OR (D) IS NOT BOUNDED, THE OTHER HAS NO FEASIBLE SOLUTION

b) IF BOTH HAVE A FEASIBLE SOLUTION THEN FOR OPTIMAL SOLUTIONS $x^*$ OF (P) AND $y^*$ OF (D) HOLDS $c^T x^* = b^T y^*$

PROOF FROM FARKAS LEMMA

ONLY b)

$\delta = c^T x^*$

$x^*$ OPTIMAL $\Rightarrow c^T x^* = b^T y$

CONSIDER $Ax \le b,\ c^T x \ge \delta\ ,\ (x \ge 0)$ $\quad \hat{A}x \le \hat{b}_0$

$\rightarrow$ HAS A NON-NEGATIVE SOLUTION $\{\hat{A}x \le \hat{b}_\varepsilon$

CONSIDER $Ax \le b,\ c^T x \ge \delta + \varepsilon \quad (\varepsilon > 0)$

$\rightarrow$ HAS NO NONNEGATIVE SOLUTION

$$\hat{A} = \begin{pmatrix} A \\ -c^T \end{pmatrix} \qquad \hat{b}_\varepsilon = \begin{pmatrix} b \\ -\delta - \varepsilon \end{pmatrix}$$

$\rightarrow$ APPLY FARKAS b) $\qquad$ $\exists$ NONNEGATIVE

$\Rightarrow \exists\ \hat{y} = (u, \underset{\in\mathbb{R}}{z}) \in \mathbb{R}^{m+1}$ ST.

$\hat{y}^T \hat{A} \ge 0^T \ \&\ \hat{y}^T \hat{b}_\varepsilon < 0$

$u^T A - z c^T \ge 0^T \ \&\ u^T b - z(\delta + \varepsilon) < 0$

$A^T u \geq z \cdot c \qquad b^T u < z(\gamma + \varepsilon)$

BUT F.L. FOR $\varepsilon = 0$ GIVES

(AS SOLUTION EXISTS) $b^T u \geq z \cdot \gamma$

SO $z \neq 0$ $(z > 0)$ $\qquad z = 0 \Rightarrow$ ↯

DEFINE $v := \frac{u}{z}$

$A^T v \geq c \quad \& \quad b^T v < \gamma + \varepsilon$

$v \geq 0$

NOTE DUAL + BOUND ON MIN.

$\forall \varepsilon \; \exists v$ S.T. $b^T v < \gamma + \varepsilon$. BY WEAK DUALITY $c^T x^* \leq b^T v \quad \forall v$

$\Rightarrow \exists y^*$ S.T. $c^T x^* = b^T y^*$ $\qquad \square$

↳ THIS SET FOR DIFF $\varepsilon$ IS CLOSED & NOT GOING TO INFINITY (BOUNDED)

# GEOMETRIC INTERPRETATION

$$
(P) \begin{cases} \text{MIN } 18x_1 + 12x_2 + 2x_3 + 6x_4 \\ \text{S.T. } 3x_1 + x_2 - 2x_3 + x_4 = 2 \\ \qquad x_1 + 3x_2 \qquad - x_4 = 2 \end{cases}
$$

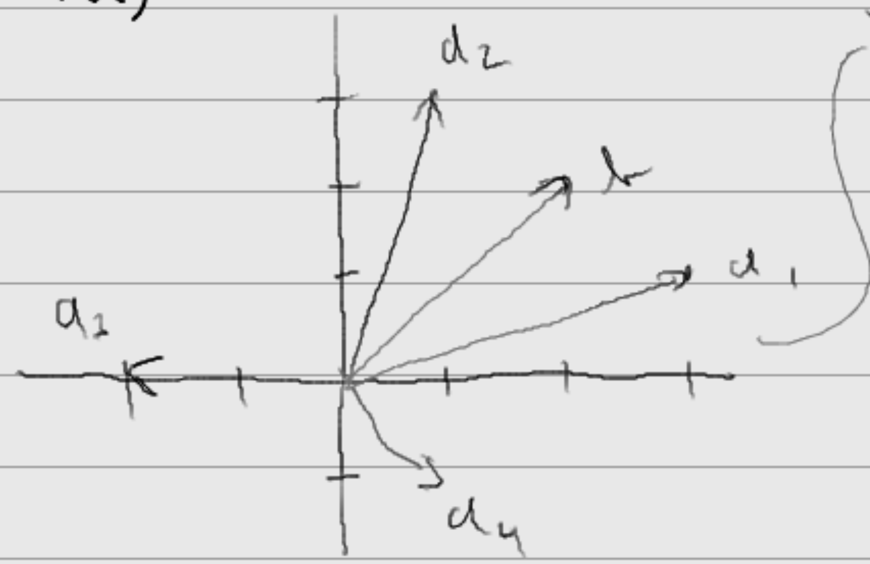
$x_1, x_2, x_3, x_4 \geq 0$

$a_1$ $a_2$

DUAL: $\max\ 2y_1 + 2y_2$

$$\left.\begin{aligned} \text{S.T.}\quad 3y_1 + y_2 &\le 18 \\ y_1 + 3y_2 &\le 12 \\ -2y_1 &\le 2 \\ y_1 - y_2 &\le 6 \end{aligned}\right\} \quad (D)$$

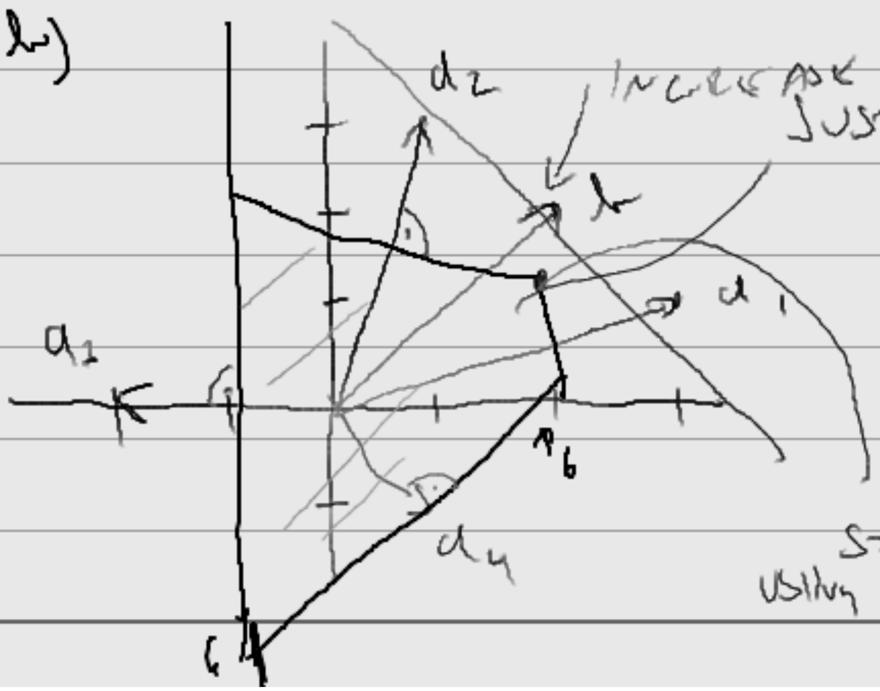
FIG (a)




SOLUTION USING 2 VECTORS

SOLUTION USING $a_1, a_2$

FIG (b)




INCREASE IN DIRECTION JUST SKETCH $a_1$

SOLUTION USING $a_1, a_2$ =

# SENSITIVITY

$\min c^T x$      $\max b^T y$

$Ax = b$      $A^T y \leq c$

$x \geq 0$

? HOW SOLUTION CHANGES IF $b$ CHANGES A LITTLE BIT ???

- WHICH OF THE CONDITION IS GOOD TO VIOLATE? (OR GUARD TO STAY?)
  ↳ PRODUCTION SCHEDULING?

SOLUTION:

EXAMPLE BEFORE

SAY $X^* = (X_B, 0)$

$A = (B \mid TRASH)$      $B = \begin{pmatrix} 3 & 1 \\ 1 & 3 \end{pmatrix}$

$B$ .. BASE FOR SOLUTION

$c^T X^* = c_B^T X_B$ ; $X_B = B^{-1} b$

$= c_B^T B^{-1} b$

$\leadsto y^{*T} = c_B^T B^{-1}$

$b \to (b + \Delta b) \leftarrow$ IF $\Delta b$ SMALL, BASE $B$ STAYS - SEE EXAMPLE.

$X_B^{+\Delta} = B^{-1}(b + \Delta b) = X_B + \Delta X_B = B^{-1} \Delta b$

CHANGE IN OBJECTIVE FUNCTION:

$$\Delta z = c_B^T \Delta x_B = c_B^T B^{-1} \Delta b = y^T \Delta b$$

SO IF $b$ CHANGED BY $\Delta b$, OBJECTIVE FUNCTION CHANGED BY $y^T \Delta b$

- $y^*$ GIVES SENSITIVITY OF THE SOLUTION

# COMPLEMENTARY SLACKNESS

$$(P)\begin{cases} \min c^T x \\ Ax = b \\ x \geq 0 \end{cases} \qquad (D)\begin{cases} \max b^T y \\ A^T y \leq c \\ \end{cases}$$

THEOREM:

LET $x$ AND $y$ BE FEASIBLE FOR (P) & (D)

$x$ AND $y$ ARE OPTIMAL PAIR IFF

a) $x_i > 0 \Rightarrow y^T a_i = c$

b) $x_i = 0 \Leftarrow y^T a_i < c$

PROOF:

$$a, b \Rightarrow (y^T A - c^T)\, x = 0$$

$$y^T A x - c^T x = 0$$

$$y^T b = c^T x \Rightarrow \text{OPTIMAL}$$

DUALITY

BACK: $\underbrace{(y^T A - c^T)}_{\leq 0} \underbrace{x}_{\geq 0} = 0$

$\Rightarrow$ $a_1$ & $b_1$ MUST HOLD

B

FOR

$$\begin{array}{ll} \text{MIN } c^T x & \text{MAX } b^T y \\ Ax \geq b & A^T y \leq c \\ x \geq 0 & y \geq 0 \end{array}$$

SLACKNESS

EX

SUPPOSE DIET PROBLEM

$a_j x > b_j \Rightarrow y_j = 0$

↳ IN OPTIMAL SOLUTION WE GET > THAN WHAT WE NEED OF $b_j$. $\Rightarrow$

$\Rightarrow$ $b_j$ IS NOT „LIMITTING" IT IS „FOR FREE"

$\Rightarrow$ COST $y_j = 0$

# SIMPLEX NOT P-TIME [KLEE - MINTY]

WE DON'T KNOW SIMPLEX METHOD, SO JUST A QUICK „IDEA"

$$(P)\begin{cases} \max\ 100x_1 + 10x_2 + x_3 \\ x_1 \leq 1 \\ 20x_1 + x_2 \leq 100 \\ 200x_1 + 20x_2 + x_3 \leq 10\,000 \\ x_1 \geq 0 \quad x_2 \geq 0 \quad x_3 \geq 0 \end{cases}$$

„BE GREEDY" — MAX THE COORDINATE WITH GREATEST COEFFICIENT IN MAX

$x_1 = x_2 = x_3 = 0$

1. $x_1 = 1 \quad x_2 = 0 \quad x_3 = 0$ — 100
2. $x_1 = 1 \quad x_2 = 80 \quad x_3 = 0$ — 900
3. $x_1 = 1 \quad x_2 = 80 \quad x_3 = 8200$ — 9100
4. $x_1 = 1 \quad x_2 = 0 \quad x_3 = 9800$ — 9900
5. $x_1 = 0 \quad x_2 = 100 \quad x_3 = 8000$ — 9000
6. $x_1 = 0 \quad x_2 = 0 \quad x_3 = 10000$ — 10000

?

IMAGE OF CUBE