NOTES:

$$\max\ c^T x \qquad \min\ y^T b$$
$$\text{s.t. } Ax \le b \qquad \text{s.t. } y^T A \ge c^T$$
$$x \ge 0 \qquad y \ge 0$$

→ FIND FEASIBLE POINT

$$c^T x \ge y^T b$$
$$Ax \le b \quad \Leftarrow \text{ NOT FULL DIMENSION?}$$
$$A^T y \ge c^T$$
$$x, y \ge 0$$

ADD + ε TO $\quad c^T x \ge y^T b - \varepsilon$

→ SOLVES UP TO ε _ GOOD ENOUGH

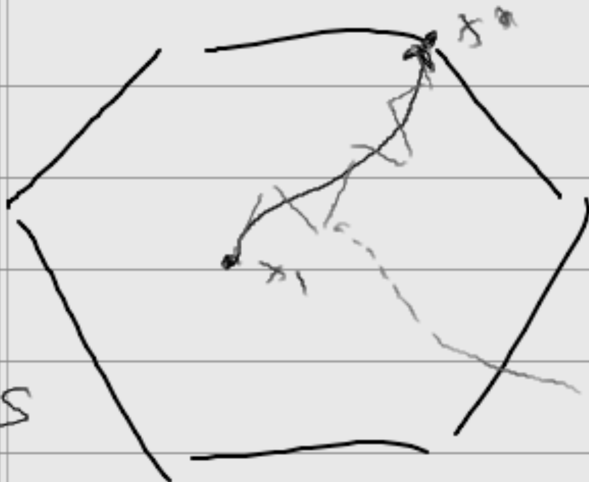
(ε MATTERS SMALL :))

LUNCH × NATIONAL BUDGET

— NICE THEORY, DOES NOT WORK

FAST IN PRACTICE :(

INTERIOR POINT METHODS



S

START INSIDE WITH x_1
MOVE TOWARDS OPTIMUM FROM INSIDE

MOVE "ALONG" PATH TO x^*

OPTIMIZE BY SMALL STEPS "LOCALLY"

$S = \{x : g_j(x) \geq 0\}$ AND
$S^\circ = \{x \quad g_j(x) > 0\}$ NONEMPTY

FOR $x \in S^\circ$:

$$\varphi(x) = -\sum_{j=1}^{m} \log g_j(x)$$

$X_1 = \text{ARGMIN } \varphi(x)$
→ ANALYTIC CENTER OF S

μ FROM ∞ TO 0 →
ITERATIONS ON μ

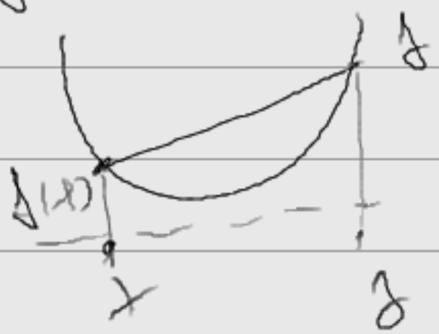
$$\text{MIN } c^T \cdot x - \mu \sum \log g_j(x)$$
s.t. ...

CONVEX FUNCTIONS

BOYD, CONVEX OPTIMIZATION, CHAPTER 3

$f : \mathbb{R}^n \to \mathbb{R}$ IS CONVEX IF dom f IS CONVEX & $\forall\, x, y \in \text{dom } f \quad \lambda \in [0,1]$

$$f(\lambda x + (1-\lambda) y) \leq \lambda f(x) + (1-\lambda) f(y)$$



- STRICTLY CONVEX IF $x \neq y$, $\lambda \in (0,1)$ & $f(\lambda x + (1-\lambda) y) < \lambda f(x) + (1-\lambda) f(y)$
- f CONCAVE IF $-f$ CONVEX
- $f(x) = ax + b$

  $f(x) = e^{ax}$

  $f(x) = -\log x \quad (x > 0)$

EXTENDED VALUE (FOR COMFORT)

DEF

$$\tilde{f}(x) = \begin{cases} f(x) & x \in \operatorname{dom} f \\ \infty & x \notin \operatorname{dom} f \end{cases}$$

→ SIMPLE NOTATIONS .. NO NEED TO CONSIDER dom f EVERY WHERE

EX $f = f_1 + f_2$

---→ dom f IS INTERSECTION OF dom f .. NEEDS CHECKING. FOR $\tilde{f}$ CLEAR.

→ EXTENDING f TO $\mathbb{R}^n$ * NOT ALWAYS HELP

FIRST DERIVATIVES

RECALL

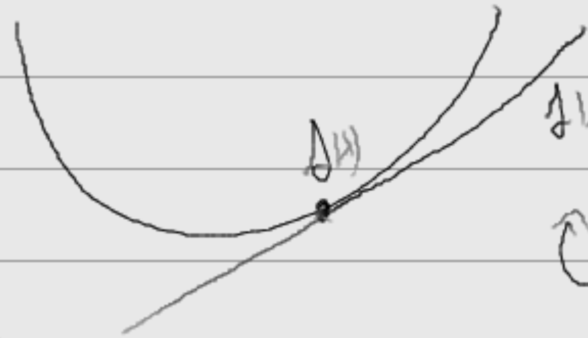
$$\nabla f(x) = \left(\frac{\partial f(x)}{\partial x_1}, \frac{\partial f(x)}{\partial x_2}, \dots \frac{\partial f(x)}{\partial x_n} \right)$$

THEOREM

f IS CONVEX IFF dom f CONVEX AND

$$f(y) \geq f(x) + \nabla f(x)^T (y - x)$$

$\forall\, x, y \in \operatorname{dom} f$



$f(x) + \nabla f(x)^T (y-x) = f(y)$

GLOBAL UNDERESTIMATOR

$\nabla f(x) = 0 \Rightarrow x$ IS GLOBAL MINIMIZER

LOCAL PROPERTY $\Rightarrow$ GLOBAL PROPERTY

SECOND DERIVATIVES

RECALL HESSIAN OF $f(x)$ IS

$$H_f(x) = \nabla^2 f(x) = \left(\frac{\partial^2 f(x)}{\partial x_1 \partial x_i} \quad \cdots \right.$$

$$\rightarrow \quad \nabla^2 f(x)_{ij} = \frac{\partial^2 f(x)}{\partial x_i \partial x_j}$$

IF CONTINUOUS SYMMETRIC

THEOREM:

- f IS CONVEX IFF $H_f(x)$ IS POSITIVE SEMIDEF
- IF $H_f(x)$ STRICTLY POSITIVE THEN f IS STRICTLY CONVEX $\forall x \in \mathrm{dom}\, f$

MATRIX $A \in \mathbb{R}^{n \times n}$ IS POSITIVE (SEMI) DEF.
IF $\forall x \in \mathbb{R}^n \setminus \{0\}$

$$x^T A x \underset{(-)}{>} 0.$$

(CONDITION -- ALL EIGENVALUES OF A ARE $\underset{(-)}{>} 0$) & SOME OTHER PROPERTIES IN A MINUTE

EXAMPLES OF CONVEX FUNCTIONS

- NORM ON $\mathbb{R}^n$
- MAX FUNCTION $f(x) = \max\{x_1 .. x_n\} = \|x\|_\infty$
- QUADRATIC OVER LINEAR
  $f(x,y) = \frac{x^2}{y}$ $y > 0,\ x \in \mathbb{R}$
- GEOMETRIC MEAN
  $f(x) = \left(\prod_{i=1}^{n} x_i\right)^{1/n}$ (CONCAVE)
- LOG-DETERMINANT USING H_f
  $f(X) = \log\det(X)$, WHERE
  X POSITIVE DEFINITE
  - - (CONCAVE)
- $x^T P x + q^T x + r$ $P \succ 0$, CONVEX H_f

NORM $f : \mathbb{R}^n \to \mathbb{R}$

$$f(\lambda x + (1-\lambda) y) \leq f(\lambda x) + f((1-\lambda) y) =$$

NORM TRIANGLE INEQUALITY

$$= \lambda f(x) + (1-\lambda) f(y)$$

BY DEF. OF NORM

FIND CONVEX BY DIRECTIONS

LOG-DETERMINANT

$f(x) = \log \det(x)$

LET $X = Z + tV$ WHERE $Z \succ 0$ AND V SYMMETRIC & $t \in \mathbb{R}$ & $X \succ 0$

$t \in$ INTERVAL ST $(Z + tV) \succ 0$

$$g(t) = \log \det(Z + tV) \qquad Z = Z^{1/2} \cdot Z^{1/2}$$

$$= \log \det\left(Z^{1/2}(I + t Z^{-1/2} V Z^{-1/2}) \cdot Z^{1/2}\right)$$

HAS INVERZ

$$= \log \det Z + \log \det\left(I + t Z^{-1/2} V Z^{-1/2}\right)$$

$\det A = \prod \lambda_i$ -- EIGENVALUES

$$= \log \det Z + \log\left(\prod (1 + t \lambda_i)\right)$$

→ WHERE λ IS FOR $Z^{-1/2} V Z^{-1/2}$

$$= \sum_{i=1}^{n} \log(1+t\lambda_i) + \log\det Z$$

$$g'(t) = \sum_{i=1}^{n} \frac{\lambda_i}{1+t\lambda_i} \qquad g''(t) = -\sum_{i=1}^{n} \frac{\lambda_i^2}{(1+t\lambda_i)^2}$$

$g''(t) < 0 \Rightarrow g(t)$ CONCAVE $\Rightarrow$

$\Rightarrow f$ CONCAVE