<u>NORM</u> $f : \mathbb{R}^n \to \mathbb{R}$

$$f(\lambda x + (1-\lambda) y) \leq f(\lambda x) + f((1-\lambda) y) =$$

NORM TRIANGLE INEQUALITY

$$= \lambda f(x) + (1-\lambda) f(y)$$

BY DEF. OF NORM

FIND CONVEX

BY DIRECTIONS

## LOG-DETERMINANT

$f(X) = \log \det(X)$

LET $X = Z + tV$ WHERE $Z > 0$ AND $V$ SYMMETRIC & $t \in \mathbb{R}$ & $X > 0$

$t \in$ INTERVALS $(Z + tV) > 0$

$$g(t) = \log \det(Z + tV) \qquad Z = Z^{1/2} \cdot Z^{1/2}$$

$$= \log \det\left(Z^{1/2}(I + t Z^{-1/2} V Z^{-1/2}) \cdot Z^{1/2}\right)$$

HAS INVERSE

$$= \log \det Z + \log \det\left(I + t Z^{-1/2} V Z^{-1/2}\right)$$

$\det A = \prod \lambda_i$ -- EIGENVALUES

$$= \log \det Z + \log\left(\prod (1 + t\lambda_i)\right)$$

$\to$ WHERE $\lambda$ IS FOR $Z^{-1/2} V Z^{-1/2}$

$$= \sum_{i=1}^{n} \log(1+t\lambda_i) + \log\det Z$$

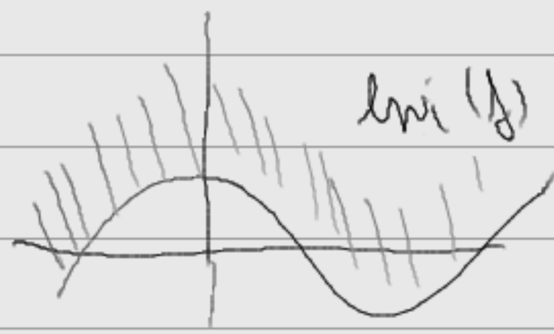
$$g'(t) = \sum_{i=1}^{n} \frac{\lambda_i}{1+t\lambda_i} \qquad g''(t) = -\sum_{i=1}^{n} \frac{\lambda_i^2}{(1+t\lambda_i)^2}$$

$g''(t) < 0 \Rightarrow g(t)$ CONCAVE $\Rightarrow$

$\Rightarrow f$ CONCAVE

<u>EPIGRAPH</u> OF $f: \mathbb{R}^n \to \mathbb{R}$ IS

$$\text{epi } f = \{(x,t) : f(x) \leq t, x \in \text{dom } f\}$$




Θ: epi(f) IS CONVEX IFF $f$ CONVEX

- GOOD FOR DECIDING CONVEXITY
- WE ALREADY KNOW

# JENSEN'S INEQUALITY

IF $f$ CONVEX THEN

- $\forall x, y \quad 0 < \lambda < 1$:

$$f(\lambda x + (1-\lambda) y) \le \lambda f(x) + (1-\lambda) f(y)$$

- $\forall x_1 \dots x_n \quad 0 < \lambda_1 \dots \lambda_n < 1, \quad \sum \lambda = 1$

$$f\left(\sum \lambda_i x\right) \le \sum \lambda f(x)$$

- $\lambda$ AS PROBABILITY ... RANDOM VAR. $Z$

$$f(E z) \le E(f(z))$$

REAL JENSEN $f\left(\frac{x+y}{2}\right) \le \frac{f(x) + f(y)}{2}$

HOLDERS $$\sum_{i=1}^{n} x_i y_i \le \left(\sum |x_i|^p\right)^{1/p} \left(\sum |y_i|^q\right)^{1/q}$$

$p > 1, \quad \frac{1}{p} + \frac{1}{q} = 1$

$f(x) = -\log(x)$

$$-\log(\lambda a + (1-\lambda) b) \le -\lambda \log a - (1-\lambda) \log b$$

$$\lambda a + (1-\lambda b) \ge a^{\lambda} b^{(1-\lambda)}$$

PUT

$$a = \frac{|x_i|^p}{\sum_{j=1} |x_j|^p} \qquad b = \frac{|y_i|^q}{\sum |y_j|^q}$$

WHERE $\frac{1}{p} = \lambda$

$$\left(\frac{|x_i|^p}{\sum_{j=1}^{n} |x_j|^p}\right)^{1/p}\left(\frac{|y_i|^q}{\sum |y_j|^q}\right)^{1/q} \le$$

$$\le \frac{|x_i|^p}{p \sum |x_j|^p} + \frac{|y_i|^q}{q \sum_{j=1} |y_j|^q}$$

SUM OVER $i$:

$$\frac{1}{\left(\sum |x_j|^p\right)^{1/p}\left(\sum |y_j|^q\right)^{1/q}} \cdot \sum |x_i|^{\frac{p}{p}} |y_i|^{\frac{q}{q}} \le \frac{1}{p} + \frac{1}{q}$$

$$\sum_{i=1}^{n} x_i y_i \le \left(\sum |x_i|^p\right)^{1/p}\left(\sum |y_i|^q\right)^{1/q}$$

$\square$

CHECKING FUNCTION IS CONVEX

- FROM DEFINITION
- USING $H_f(x) \geq 0$
- COMPOSITION OF SIMPLE FUNCTIONS
  - $f$ CONVEX $\Rightarrow \alpha f$ CONVEX $(\alpha \geq 0)$
  - $f_1, f_2$ CONVEX $\Rightarrow f_1 + f_2$ CONVEX
    - ↳ NON-NEGATIVE WEIGHTED SUMS
  - $f$ CONVEX $\Rightarrow f(Ax+b)$ IS CONVEX

    $\varphi(x) = -\sum \log(b_i - a_i^T x)$

    WHERE $x \in \{x \mid a_i^T x < b_i \ \forall i \in 1..m\}$

    ? FUNCTION FOR INTERIOR POINT METHOD
  - COMPOSITION

    $f(x) = h(g(x))$

    $f'(x) = h'(g(x))\, g'(x)$

    $f''(x) = h''(g(x)) \cdot g'(x)^2 + h'(g(x))\, g''(x)$

    $? \geq 0$ ?

$h(g(x))$ $g$ CONVEX, $h$ CONVEX, NONDECREASING $\Rightarrow$

$\Rightarrow f$ CONVEX

$1/g(x)$ $g$ CONCAVE, $h$ CONVEX, NONINCREASING

- POINTWISE MAXIMUM

$f_1 \dots f_k$ CONVEX THEN

$$f(x) = \max_i f_i(x)$$

EX

- $f(x) = \max_i \left(a_i^T x + b_i\right)$ ... PIECEWISE LINEAR FUNCTION

- SUM OF $k$ LARGEST COMPONENTS

$$f(x) = x_{[1]} + x_{[2]} + \dots x_{[k]}$$

$$f(x) = \max\{x_{i_1} + \dots + x_{i_k} \mid 1 \le i_1 < \dots \le i_k\}$$

- POINTWISE SUPREMUM

IF $f(x,y)$ IS CONVEX IN $x$ FOR EVERY $y \in A$

THEN

$$g(x) = \sup_{y \in A} f(x,y) \quad \text{IS CONVEX}$$

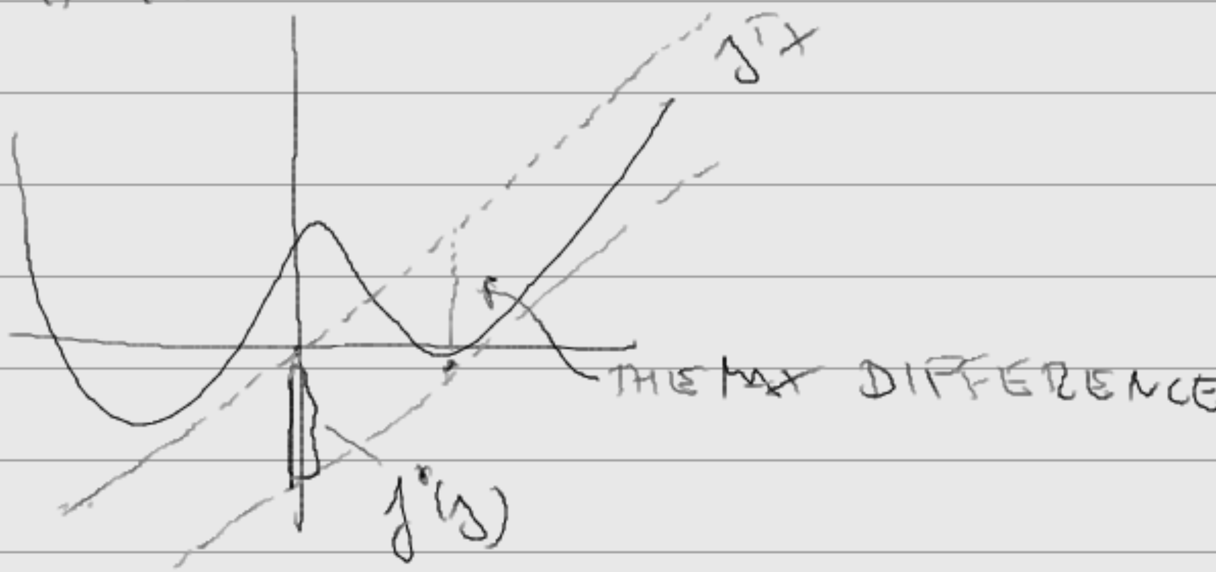
CONJUGATE FUNCTION

$f: \mathbb{R}^n \to \mathbb{R}$, THEN $f^*(y): \mathbb{R}^n \to \mathbb{R}$

$$f^*(y) = \sup_{x \in \mathrm{dom} f} \left(y^T x - f(x)\right)$$

- FOR FIXED $x$ CONVEX $\Rightarrow f^*$ IS CONVEX
(EVEN IF $f$ NOT!)

IMG:




Ex $f(x) = -\log(x) \qquad x > 0$

$$f^*(y) = \sup_{x>0} (xy + \log x)$$

$\Rightarrow y < 0$ ONLY

FIX $y$.. $g(x) = xy + \log x)$

$g'(x) = y + \frac{1}{x} = 0$

$$x = -\frac{1}{y}$$

$\Rightarrow \quad f^*(y) = -1 - \log(-y)$ $\quad y < 0$

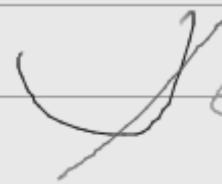
$+\infty$ OTHER

Ex $\quad f(x) = \left(\frac{1}{2}\right) x^T Q x \quad Q \succ 0$

$$f^*(y) = \sup_x \left( y^T x - \left(\tfrac{1}{2}\right) x^T Q x \right)$$

$$= \frac{1}{2} y^T Q^{-1} y$$

LIKE



← AFFINE

IF $f$ CONVEX & $f$ CLOSED ( $\text{epi}(f)$ IS CLOSED )

THEN $f^{**} = f$

PROOF MAYBE HW

FENCHEL'S INEQUALITY

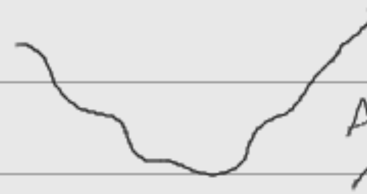
$$f(x) + f^*(y) \geq x^T y \quad \text{.. FROM DEF :)}$$

$$x^T y \leq \left(\tfrac{1}{2}\right) x^T Q x + \left(\tfrac{1}{2}\right) y^T Q^{-1} y$$

# QUASICONVEX FUNCTIONS (3.4)

IDEA

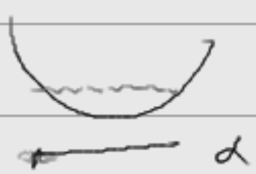
← NICE

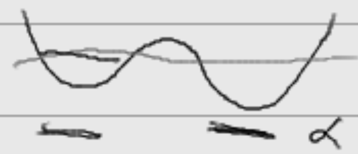


ALSO NICE ENOUGH

$\alpha$- SUBLEVEL SET OF $f$

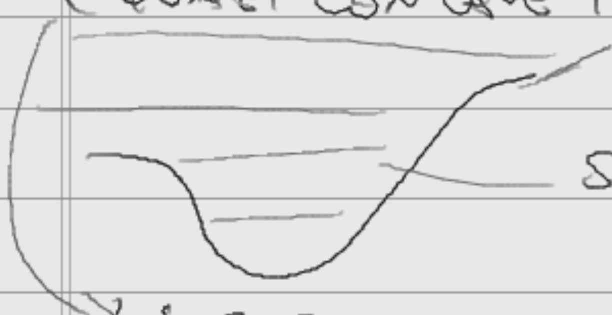
$$S_\alpha = \{x \in \mathrm{dom}\, f : f(x) \le \alpha\}$$




$f$ IS QUASICONVEX IF $\forall_\alpha$ $S_\alpha$ IS CONVEX

(QUASI CONCAVE IF $-f$ QUASICONVEX



STILL CONVEX

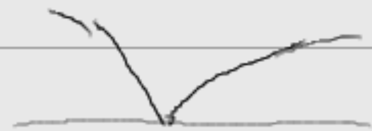
$\alpha$ SUPERLEVEL SETS ... $f(x) \ge \alpha$

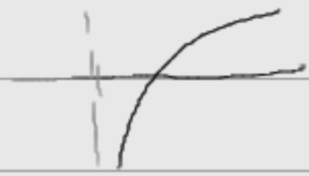
$f$ IS QUASILINEAR IF QUASI CONVEX &

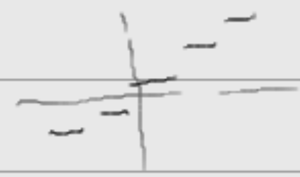
& QUASICONCAVE

$\{x \in \mathrm{dom}\, f : f(x) = \alpha\}$ ARE CONVEX

Ex

$\sqrt{|x|}$  ... QUASICONVEX

$\log(x)$, $x>0$  ... QUASILINEAR

$\lceil x \rceil$  QUASILINEAR

$$f(x) = \frac{a^T x + b}{c^T x + d} \qquad \mathrm{dom}\, f = \{x \mid c^T x + d > 0\}$$

$$S_\alpha = \{x \mid c^T x + d > 0,\ a^T x + b \le \alpha(c^T x + d)\}$$

INTERSECTION OF 2 CONVEX SETS = cvx.

→ ALSO QUASILINEAR

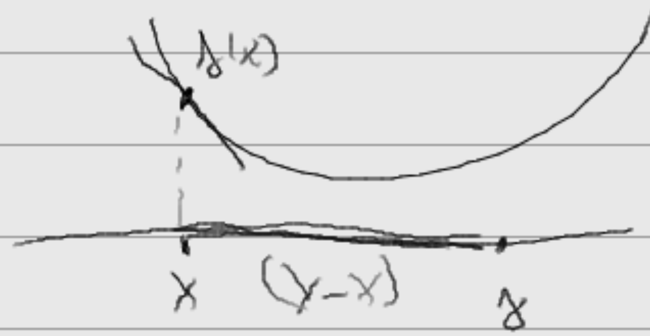
- "JENSEN" $f$ QUASICONVEX IFF 

  $\lambda \in (0,1)$ .. $f(\lambda x + (1-\lambda) y) \le \max\{f(x), f(y)\}$

- $f$ QUASICONVEX IFF $\mathrm{dom}\, f$ convex &

  & $f(y) \le f(x) \Rightarrow \nabla f(x)^T (y - x) \le 0$

  HALFSPACE IN $y$



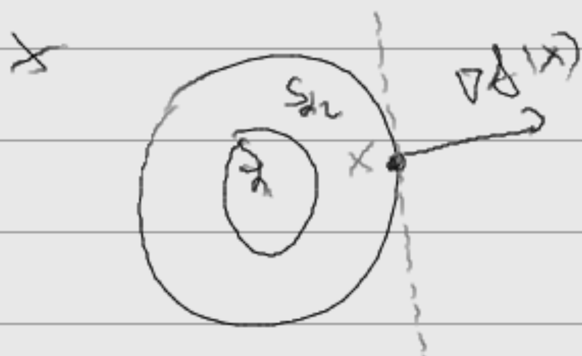

$f'(x) < 0$

$\Rightarrow f(y) < f(x) \Rightarrow y > x$

$f'(x) > 0 \Rightarrow$

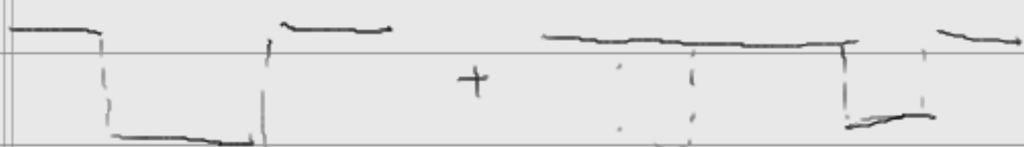
$\Rightarrow f(y) < f(x) \Rightarrow x > y$

IN HIGHER DIMENSION

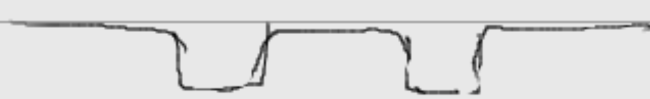



IS SUM OF 2 QUASICONVEX FUNCTIONS QUASICONVEX

NO



$\Rightarrow$

$=$  $\Leftarrow$ NOT QUASICONVEX

$f(x) = \underbrace{\max\{w_i f_i\}}_{\text{works}}$ $w_i \geq 0$ $f_i$ QUASICONV