QUASICONVEX FUNCTIONS (3.4)

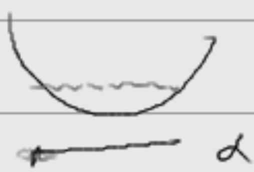
IDEA

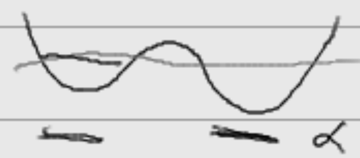
← NICE

ALSO NICE ENOUGH

α-SUBLEVEL SET OF f

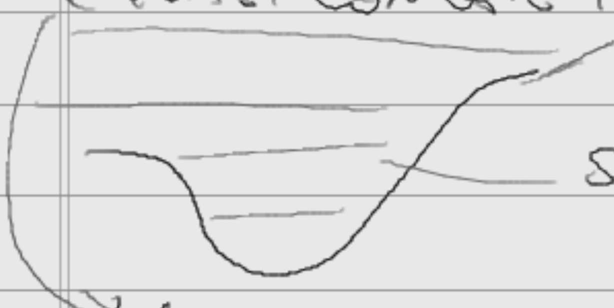
$$S_\alpha = \{x \in \mathrm{dom}\, f : f(x) \le \alpha\}$$





f IS QUASICONVEX IF $\forall \alpha$ S_α IS CONVEX

(QUASICONCAVE IF $-f$ QUASICONVEX



STILL CONVEX

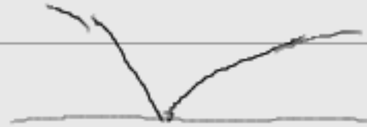
α SUPERLEVEL SETS ..$f(x) \ge \alpha$

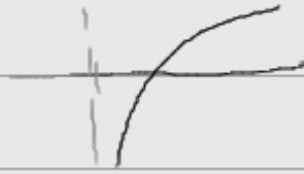
f IS QUASILINEAR IF QUASICONVEX &

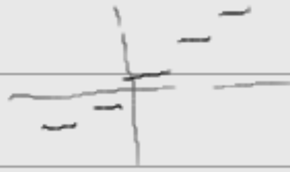
& QUASICONCAVE

$\{x \in \mathrm{dom}\, f : f(x) = \alpha\}$ ARE CONVEX

EX

$\sqrt{|x|}$  ... QUAZICONVEX

$\log(x)$, $x>0$  ... QUAZILINEAR

$\lceil x \rceil$  QUAZILINEAR

$$f(x) = \frac{a^T x + b}{c^T x + d} \qquad \text{dom } f = \{x \mid c^T x + d > 0\}$$

$$S_\alpha = \{x \mid c^T x + d > 0,\ a^T x + b \le \alpha(c^T x + d)\}$$

INTERSECTION OF 2 CONVEX SETS = dom.

→ ALSO QUAZILINEAR

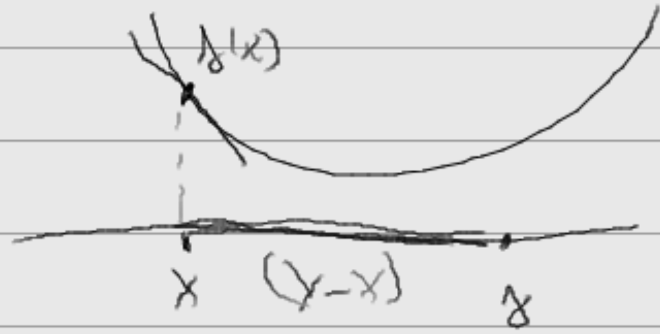
- "JENSEN" f QUAZICONVEX IFF 

  $\lambda \in (0,1)$ .. $f(\lambda x + (1-\lambda) y) \le \max\{f(x), f(y)\}$

- f QUAZICONVEX IFF dom f convex &

  & $f(y) \le f(x) \Rightarrow \nabla f(x)^T (y-x) \le 0$

  HALFSPACE IN y



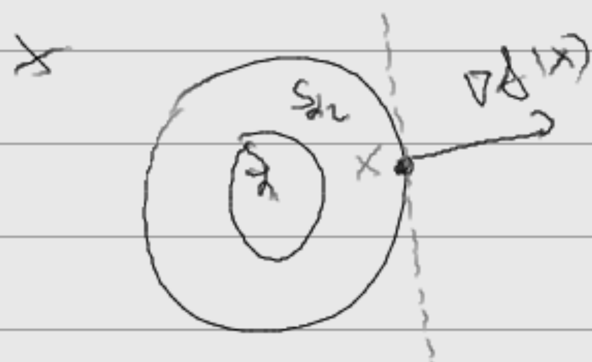
$f'(x) < 0$

$\Rightarrow f(y) \le f(x) \Rightarrow y \ge x$

$f'(x) > 0 \Rightarrow$

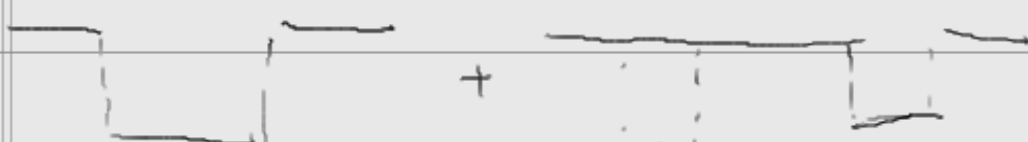
$\Rightarrow f(y) \le f(x) \Rightarrow x \ge y$

IN HIGHER DIMENSION

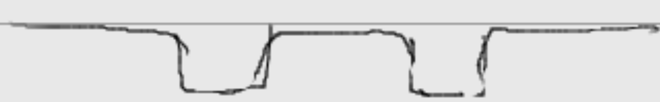


IS SUM OF 2 QUASICONVEX FUNCTIONS QUASICONVEX

NO



$\Rightarrow$

$=$  $\Leftarrow$ NOT QUASICONVEX

$f(x) = \underset{\text{works}}{\max}\{w_i f_i\}$ $\quad w_i \ge 0$ $\quad f_i$ QUASICONV

QUASICONVEX FUNCTIONS AS A CONVEX

FIND FAMILY $\phi_t : \mathbb{R}^n \to \mathbb{R}$ $\quad t \in \mathbb{R}$
SUCH THAT

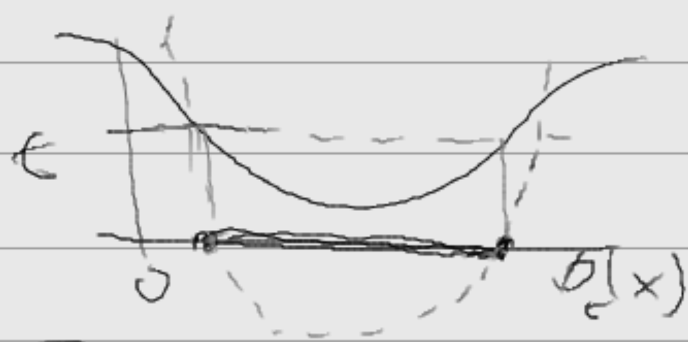
$$f(x) \le t \iff \phi_t(x) \le 0$$

t-SUBLEVEL OF $f(x)$ IS 0-SUBLEVEL OF $\phi_t(x)$

ALWAYS EXIST:

$$\phi_t(x) = \begin{cases} 0 & f(x) \le t \\ \infty & \text{OTHERWISE} \end{cases}$$

$\to$ NOT UNIQUE, FIND NICE $\phi_t(x)$
(DIFFERENTIABLE, ...)



$\phi_t(x) \le 0 \Rightarrow$
$\phi_s(x) \le 0$ IF
$s > t$

EX $\quad p$ CONVEX, q CONCAVE, $p(x) > 0 \quad q(x) > 0$

$f(x) = \dfrac{p(x)}{q(x)}$ IS QUASICONVEX

$f(x) \le t \iff p(x) - t\,q(x) \le 0$

$\phi_t(x) = p(x) - t\,q(x) \quad \in$ CONV & DECREASING IN t ✓

CONVEX OPTIM. PROBLEMS

$$
\left.\begin{aligned}
\text{MIN}\quad & f_0(x) \\
\text{ST}\quad & f_i(x) \leq 0 \quad i = 1 \ldots m \\
& a_i^T x = b_i \quad i = 1 \ldots p
\end{aligned}\right\} \text{CONV. (C) PROGRAM}
$$

$f_0, f_1, \ldots, f_m$ CONVEX

FEASIBILITY PROBLEM

FIND x

$$
\begin{aligned}
\text{S.T.}\quad & f_i(x) \leq 0 \quad i = 1 \ldots m \\
& a_i^T x = b_i \quad i = 1 \ldots p
\end{aligned}
$$

(SPECIAL CASE OF $f_0(x) = 0$)

SOLUTION TO IS FEASIBLE SET.

FEASIBLE SET OF (C) IS CONVEX

IF f_i "NICE", CAN BE SOLVED.

EX MAKING NICER

$$\left.\begin{aligned} \min\ & f_0(x) = x_1^2 + x_2^2 \\ \text{s.t.}\ & f_1(x) = x_1/(1+x_2^2) \leq 0 \\ & h_1(x) = (x_1+x_2)^2 = 0 \end{aligned}\right\}(P)$$

f_1 NOT CONVEX; h_1 NOT LINEAR

$\to f_1 \ldots x_1 \leq 0$

$$\begin{aligned} \min\ & x_1^2 + x_2^2 \\ \text{s.t.}\ & x_1 \leq 0 \\ & x_1 + x_2 = 0 \end{aligned}$$

x^* FOR SOLUTION

$\to$ EVEN OBVIOUS $x_1^* = x_2^* = 0$ SOLUTION

THEOREM

EVERY LOCAL MINIMIZER OF (C) IS OPTIMAL SOLUTION OF (C)

PROOF

x^*, δ S.T. $\forall x \in B(x^*, \delta) : f_0(x^*) \leq f(x)$

y FEASIBLE

PICK λ ST $\lambda x^* + (1-\lambda) y \in B(x^*, \delta)$

$f_0(x^*) \leq f(\lambda x^* + (1-\lambda) y) \leq \lambda f(x^*) + (1-\lambda) f_0(y)$

$$(1-\lambda) f_0(x^*) \le (1-\lambda) f(y)$$

$\Rightarrow x^*$ IS GLOBAL MINIMIZER.

(BOTH CONVEXITY OF f_0 & FEAS. SET USED)

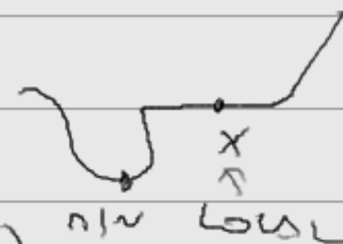
$\square$

QUASICONVEX OPTIMIZATION

$$\left.\begin{array}{ll} \min & f_0(x) \\ \text{s.t.} & f_i(x) \le 0 \\ & Ax = 0 \end{array}\right\} \text{(QC)}$$

f_i .... CONVEX !

f_0 ____ QUASICONVEX



$\Rightarrow$ LOCAL OPT. IS NOT GLOBAL

ALGORITHM: (USING CONVEX FEASIB.)

- KNOW $l \le f_0(x^*) \le u$ ← BOUNDS
- SOLVE

$$\left.\begin{array}{l} \phi_{\frac{l+u}{2}}(x) \le 0 \\ f_i \le 0 \\ Ax = 0 \end{array}\right\} \text{CONVEX} \qquad \boxed{\text{BISECTION METHOD}}$$

UPDATE l OR u AND REPEAT

SPECIAL CONVEX PROGRAMS

- LINEAR PROGRAM

$$\min\ c^T x$$
$$\text{st}\quad Gx \le h$$
$$Ax = b$$

– SURELY CONVEX :-)

- QUADRATIC PROGRAM

$$\min\ (1/2)\, x^T P x + q^T x + r$$
$$\text{st}\quad Gx \le h$$
$$Ax = b$$

WHERE P POSITIVE SEMIDEFINITE

- QUADRATICALLY CONSTRAINED QUADRATIC PROGRAM (QCQP)

$$\min\ (1/2)\, x^T P_0 x + q_0^T x + r_0$$
$$\text{st.}\ (1/2)\, x^T P_i x + q_i^T x + r_i \le 0 \quad i = 1 \ldots m$$
$$Ax = b$$

P_0 .. POS. SEMIDEF

P_i . POSITIVE DEFINITE ! ELLIPSOIDS