- GEOMETRIC PROGRAMMING

MONOMIAL FUNCTION

$$f(x) = c x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_n^{\alpha_n}$$

WHERE $c > 0$, $\alpha_i \in \mathbb{R}$, $x > 0$

POSYNOMIAL FUNCTION

$$f(x) = \sum_{k=1}^{K} c_k x_1^{\alpha_{1k}} x_2^{\alpha_{2k}} \cdots x_n^{\alpha_{nk}}$$

SUM OF MONOMIALS

$$(GP) \begin{cases} \min & f_0(x) \\ \text{S.T.} & f_i(x) \le 1 \\ & h_i(x) = 1 \end{cases}$$

$f_i(x)$ POSYNOMIAL, $h_i(x)$ MONOMIAL
$x > 0$

NOTE:

- $f_1(x)$ $f_2(x)$ MONOMIALS -- $\frac{f_1(x)}{f_2(x)}$ IS MONOMIAL

  - CONSTRAINTS $h_i(x) = h_j(x)$

- $f_1(x)$ POSYNOMIAL $f_2(x)$ MONOMIAL
  $\frac{f_1(x)}{f_2(x)}$ IS POSYNOMIAL - $f_i(x) \le h_i(x)$

IS NOT CONVEX :-(MAKING

CONVEX:

- $y_i = \log x_i$ VERY NICE :)
- TAKE log OF f_i

FOR MONOMIAL $f(x)$

$$\log f(e^y) = \log c + \alpha_1 y_1 + \alpha_2 y_2 + \dots \alpha_n y_n$$

$$GP \begin{cases} \text{MIN } \log f(e^y) \\ \text{S.T. } \log f_i(e^y) \le 0 \\ \quad\quad \log h_i(e^y) = 0 \end{cases}$$

BUT CONVEX.

SOLVABLE WITH INTERIOR POINT METHODS

GENERALIZATION:

GENERALIZED POSYNOMIAL $g(x)$

FROM (NEW) POLYNOMIALS $f_1(x), \dots$

- SUM $\sum f_i(x)$
- KOEF $\alpha f_i(x)$
- POWER $f_i(x)^3$ $f_i(x)^{2.5}$

- TAKING MAX($f_1(x)$ $f_2(x)$ ...)

EX

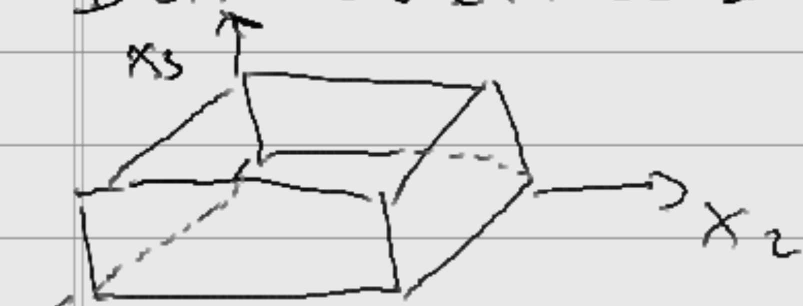
$$g(x) = (1+\max\{x_1, x_2\})\Big(\max\{1+x_1,\ x_2^{0.2} x_3^{-3.9}\} +$$
$$+ \left(0.1\, x_3 x_1^{-2} + x_2^{0.7} x_1^{1.7}\right)^{1.5}\Big)^{1.7}$$

$G(y) = \log g(e^y)$ IS CONVEX! → REDUCES TO GP.

HW

EXAMPLE:

BOX - SURFACE & VOLUME



$$V = x_1 x_2 x_3$$

$$S = x_1 x_2 + 2x_1 x_3 + 2 x_2 x_3$$

a) FIXED AMMOUNT OF MATERIAL ($S = S_0$ FIXED) & GOAL IS TO MAXIMIZE V.

MAXIMIZE $V(x_1, x_2, x_3) = x_1 x_2 x_3$

SUBJECT TO $S = S_0$

$$x_1 x_2 + 2x_2 x_3 + 2x_2 x_3 \leq S_0$$

WE WANT TO GET A CORRESPONDENCE BETWEEN S & V

$$S_0 = x_1 x_2 + 2x_1 x_3 + 2x_2 x_3 = 3\left(\frac{x_1 x_2 + 2x_1 x_3 + 2x_2 x_3}{3}\right)$$

(A-G) $$\rightarrow \geq 3\left((x_1 x_2)^{1/3} (2x_1 x_3)^{1/3} (2x_2 x_3)^{1/3}\right)$$

$$= 3 . 4^{1/3} \left(x_1^2 x_2^2 x_3^2\right)^{1/3}$$

$$= 3 . 4^{1/3} V^{2/3}$$

$\Rightarrow$ V LARGEST IF = INSTEAD OF $\geq$

OPT SOLUTION x_1^* x_2^* x_3^*

$$x_1^* x_2^* = 2x_1^* x_3^* = 2x_2^* x_3^* = \frac{S_0}{3}$$

IF

$$\Downarrow \qquad \Downarrow$$

$$x_2^* = 2x_3^* \qquad x_1^* = x_2^*$$

$$\Downarrow$$

$$x_1^* = x_2^* = \sqrt{\frac{S_0}{3}} \qquad x_3^* = \frac{1}{2}\sqrt{\frac{S_0}{3}}$$

$$\Rightarrow V = \sqrt{\frac{S_0}{3}} . \sqrt{\frac{S_0}{3}} . \frac{1}{2}\sqrt{\frac{S_0}{3}}$$

$$= \frac{S_0^{3/2}}{2 . 3^{3/2}}$$

A-G

$$\sum \delta_i x_i \geq \prod x_i^{\delta_i}$$

$$\sum \delta_i = 1$$

SEMIDEFINITE PROGRAMMING

DEFINITION

LET $A \in \mathbb{R}^{m \times m}$. THE TRACE OF A IS

$$tr(A) = \sum_{i=1}^{m} a_{ii}$$

$$SYM_m = \{ X \in \mathbb{R}^{m \times m} : X_{ij} = X_{ji} \ \forall i,j \}$$

SYMMETRIC MATRICES $m \times m$

RECALL (LP)

$$(LP) \begin{cases} \text{MAX} & c^T x \\ \text{S.T.} & Ax = b \\ & x \geq 0 \end{cases}$$

$x = (x_1 \ldots x_n)$, $c \in \mathbb{R}^n$, $b \in \mathbb{R}^m$, $A \in \mathbb{R}^{m \times n}$

$$(LP)\begin{cases} \text{MAX} & c \cdot x \\ \text{S.T.} & a_1 \cdot x = b_1 \\ & a_2 \cdot x = b_2 \\ & \vdots \\ & a_m \cdot x = b_m \\ & x \geq 0 \end{cases}$$

a_i .. i-TH ROW OF A

$c, x, a_1 \ldots a_m$ ARE VECTORS $\in \mathbb{R}^n$

$b_1, b_2, \ldots b_m \in \mathbb{R}$

REPLACE • VECTOR SPACE $\mathbb{R}^n$ BY VECTOR SPACE SYM_n

- THE DOT PRODUCT BY

$$X \cdot Y = \langle X | Y \rangle = \mathrm{Tr}(X^T Y) =$$

$$= \sum_{i=1}^{n} \sum_{j=1}^{n} X_{ij} Y_{ij}$$

- $x \geq 0$ BY X IS POSITIVE SEMIDEFINITE. DENOTED BY

$$X \succeq 0$$

($X \succ 0$, X IS POSITIVE DEFINITE)

$$(SDP)\begin{cases} \text{MAX} & Tr(C^T X) \\ \text{ST.} & Tr(A_1^T X) = b_1 \\ & \vdots \\ & Tr(A_m^T X) = b_m \\ & X \succeq 0 \end{cases}$$

$C, A_1 \ldots A_m, X \in SYM_m$

$b_1 \ldots b_m \in \mathbb{R}$

$$Tr\left(\begin{pmatrix} C_{11} & C_{12} \\ C_{12} & C_{22} \end{pmatrix}^T \begin{pmatrix} X_{11} & X_{12} \\ X_{12} & X_{22} \end{pmatrix}\right) = C_{11}X_{11} + 2C_{12}X_{12} + C_{22}X_{22}$$

→ EQUIVALENT FORM OF (SDP)

$$(SDP)\begin{cases} \text{MAX} & \sum_{i \le j} c_{ij} X_{ij} \\ \text{ST.} & \sum_{i \le j} a_{ijk} X_{ij} = b_k, \quad k = 1 \ldots m \\ & X \succeq 0 \end{cases}$$

(LP) IS SPECIAL CASE OF (SDP)

$$c \longrightarrow \begin{pmatrix} c_1 & & & 0 \\ & c_2 & & \\ & & \ddots & \\ 0 & & & c_m \end{pmatrix} = C$$

$$a_i \longrightarrow \begin{pmatrix} a_{i1} & & & 0 \\ & a_{i2} & & \\ & & \ddots & \\ 0 & & & a_{im} \end{pmatrix} = A_i$$

$$x \longrightarrow \begin{pmatrix} x_1 & & & 0 \\ & x_2 & & \\ & & \ddots & \\ 0 & & & x_m \end{pmatrix} = X$$

<u>NOTE:</u> $x \geq 0 \iff X \succeq 0$

(x_i ARE EIGENVALUES OF X)

<u>NOTE</u> HOW TO HANDLE INEQUALITY?

$$A_i \bullet X \geq b_i$$

EXTEND X:

$$\begin{pmatrix} A_i & 0 \\ 0 & -1 \end{pmatrix} \cdot \begin{pmatrix} X & 0 \\ 0 & x_i' \end{pmatrix} = b_i$$

AS x_i' IS ANYTHING ≥ 0

DUAL FORM:

(DSDP)
$$\begin{cases} \text{MINIMIZE} \quad b^T \cdot y \\ \text{S.T.} \\ \quad y_1 \cdot A_1 + y_2 A_2 + \dots y_n A_n - C \succeq 0 \end{cases}$$

$C, A_1 \dots A_n$ SYMMETRIC