DEFINITION:

LET $A \subseteq \mathbb{R}^m$. POINT $x \in A$ IS AN INTERIOR POINT OF A IF $\exists\, r > 0$ S.T. $B(x, r) \subseteq A$

THE INTERIOR A° OF A IS THE SET OF ALL INTERIOR POINTS.

LEMMA 43 – ACCESSIBILITY LEMMA (5.1.8)

LET $C \subseteq \mathbb{R}^n$ BE CONVEX SET, $C^\circ \neq \emptyset$.

IF $x \in \mathring{C}$ AND $y \in \overline{C}$ THEN

$$[x, y) = \{\lambda x + (1-\lambda) y : 0 < \lambda \leq 1\} \subseteq C^\circ$$

– PICTURE CLEAR BUT REAL PROOF TECHNICAL

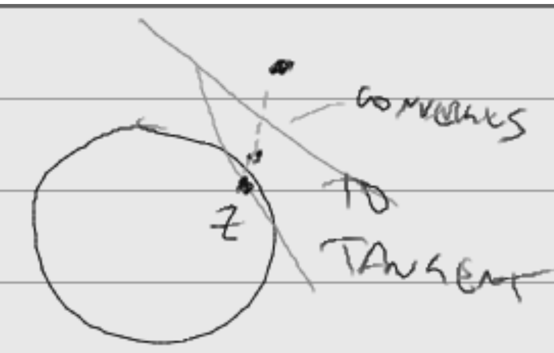
THEOREM 44 – SUPPORT THEOREM (5.1.9)

LET $C \subseteq \mathbb{R}^n$ BE CONVEX WITH NONEMPTY INTERIOR AND z BE A BOUNDARY POINT OF C. THEN $\exists\, a \in \mathbb{R}^n$, $a \neq 0$ SUCH THAT

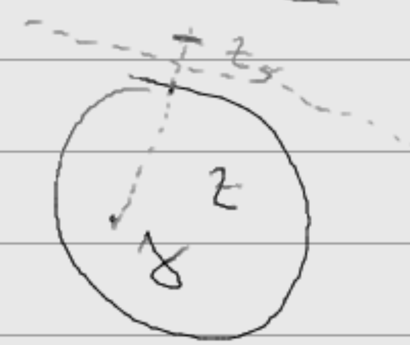
$a . x \leq a . z \quad \forall x \in C$.

$ax = a \cdot z$

IDEA



PROOF



LET $y \in C^0$. $\forall s > 1$ LET

$$z_s = y + s(z - y)$$

CL: $z_s \notin \bar{C}$ FOR $s > 1$

IF $z_s \in \bar{C}$

ACCESS LEM. $\Longrightarrow$

$$[y, z_s) = \{ y + t(z-y) : 0 \le t < s \} \subseteq C^0$$

IN PARTICULAR $z = y + (z - y) \in C^0$ ↯ AS

$z \in C \setminus C^0$

$$z_s \notin \bar{C} \overset{T.40}{\underset{\text{SEPARATION}}{\Longrightarrow}} \exists b_s : b_s \cdot x < b_s \cdot z_s \quad \forall x \in C$$

$$\frac{b_s \cdot x}{\|b_s\|} \le \frac{b_s \cdot z_s}{\|b_s\|}$$

SO WITHOUT LOSS OF GENERALITY $\|b_s\| = 1$

LET $\{s_k\}$ BE SEQUENCE $s_k > 1$, $\lim_{k \to \infty} s_k = 1$

LET $a_k = b_{s_k}$, NOTE $\|a_k\| = 1$ $\forall k$

BOLZANO - WEIERSTRASS $\Rightarrow \exists$ CONVERGING SUBSEQUENCE $\{a_{k_p}\}$ & $\lim_{k_p \to \infty} a_{k_p} = a$.

MOREOVER, $1 = \lim \|a_{k_p}\| = \|a\| \Rightarrow a \neq 0$

& $\forall x \in C$:

$$a \cdot x = \lim_{k_p} (a_{k_p} \cdot x) \leq \lim_{k_p} (a_{k_p} \cdot z_{s_{k_p}}) = a \cdot z$$

$\square$

THEOREM 21 (2.3.5)

LET $C \subseteq \mathbb{R}^n$ BE CONVEX AND $f\ C \to \mathbb{R}$ HAS CONTINUOUS ∂

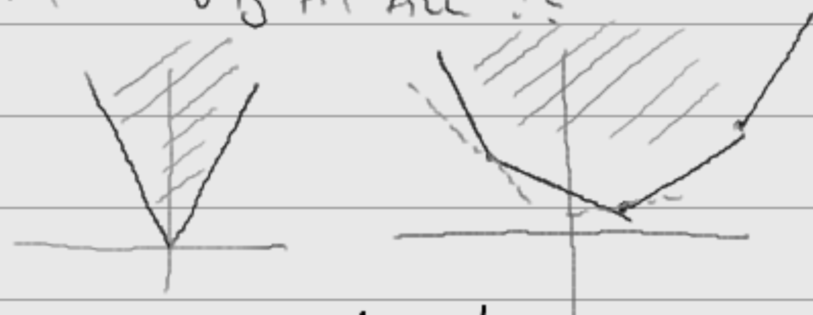
a) f IS CONVEX ON C IFF $\forall x, y \in C$

$$f(x) + \nabla f(x)(y - x) \leq f(y)$$

b) f IS STRICTLY CONVEX ON C IFF $\forall x, y \in C$, $x \neq y$

$$f(x) + \nabla f(x)(y - x) \leq f(y)$$

WHAT IF f DOES NOT HAVE CONTINUOUS ∂f?
OF NO ∂f AT ALL ??

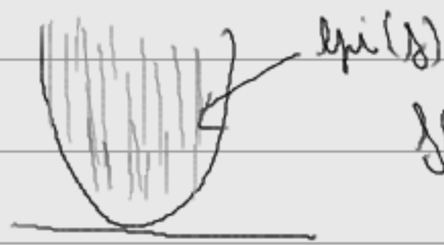


THEOREM 45 (21 / 2.3.5 VARIANT - 5.1.10)
LET $C \subseteq \mathbb{R}^n$ BE CONVEX WITH NONEMPTY INTERIOR IN $\mathbb{R}^n$. LET $f: C \to \mathbb{R}$ BE CONVEX
IF $x^0 \in C^0$ THEN $\exists\, d \in \mathbb{R}^n$ S.T.

$$f(x) \geq f(x^0) + d \cdot (x - x^0) \qquad \forall x \in C$$

DEF: LET $C \subseteq \mathbb{R}^n$ BE CONVEX
$f: C \to \mathbb{R}$ EPIGRAPH $\mathrm{epi}(f)$ IS

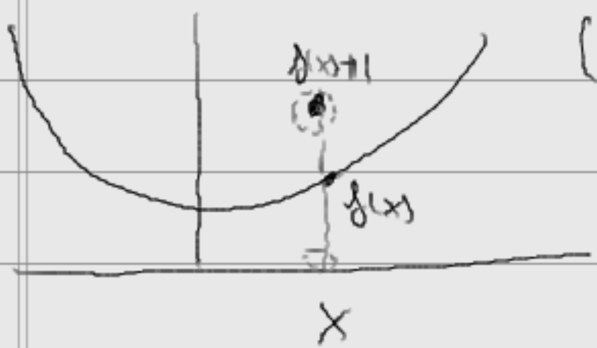
$$\mathrm{epi}(f) = \{(x, r) \in \mathbb{R}^{n+1},\ x \in C,\ r \in \mathbb{R},\ r \geq f(x)\}$$



$f(x) = x^2$
$x \in \mathbb{R}$

- f IS CONVEX IFF $epi(f)$ IS CONVEX (HW)

- IF f CONVEX AND C HAS INTERIOR POINTS IN $\mathbb{R}^m$
  - $epi(f)$ HAS INTERIOR POINTS IN $\mathbb{R}^{m+1}$
    $x \in C^\circ \Rightarrow (x, f(x)+1) \in epi(f)^\circ$
  - $x \in C^\circ \Rightarrow (x, f(x)) \in epi(f)$ BUT
    $(x, f(x)) \notin epi(f)^\circ$



DEF

VECTOR d FROM THM 45 IS CALLED **SUBGRADIENT** OF f AT x°

SET $X = \{d \in \mathbb{R}^m : f(x) \geq f(x^\circ) + d \cdot (x - x^\circ) \ \forall x \in C\}$

IS CALLED **SUBDIFFERENTIAL** OF f AT x°.

- IF $\nabla f(x^\circ)$ EXISTS THEN $X = \{\nabla f(x^\circ)\}$

5.2. KARUSH-KUHN-TUCKER-THEOREM

DEF LET $C \subseteq \mathbb{R}^n$ AND

$f, g_1, \ldots g_m : C \to \mathbb{R}$. THEN

THE FOLLOWING IS A PROGRAM:

$$(P) \begin{cases} \text{MINIMIZE } f(x) \\ \text{SUBJECT TO } g_1(x) \leq 0, \ldots g_m(x) \leq 0 \\ \text{WHERE } x \in C \end{cases}$$

- $f(x)$ IS OBJECTIVE FUNCTION
- $g_1(x) \leq 0, \ldots g_m(x) \leq 0$ ARE (INEQUALITY) CONSTRAINTS
- $x \in C$ IS A FEASIBLE POINT FOR (P) IF IT SATISFIES ALL CONSTRAINTS
- FEASIBILITY REGION FOR (P) IS THE SET OF ALL FEASIBLE POINTS OF (P) (DENOTED F)
- IF FEASIBLE REGION $\neq \emptyset$, THEN (P) IS CONSISTENT
- LET x BE FEASIBLE POINT FOR (P) SUCH THAT $\forall i \quad g_i(x) < 0$ THEN (P) IS CALLED SUPERCONSISTENT AND x IS CALLED A SLATER POINT FOR (P).

- IF x^* IS FEASIBLE POINT SUCH THAT $f(x^*) \le f(x)$ $\forall x$ FEASIBLE FOR (P) THEN x^* IS A **SOLUTION** FOR (P).
- IF $f, g_1 \ldots g_m$ AND C ARE CONVEX THEN (P) IS CALLED **CONVEX PROGRAM**

IF (P) IS CONVEX THEN THE FEASIBILITY REGION F FOR (P) IS A CONVEX SET.

$$F = \bigcap_{i=1}^{m} G_i \qquad G_i = \{x \in C : g_i(x) \le 0\}$$

RECALL

- SMALLEST $\beta \in \mathbb{R}$ ST $f(x) \le \beta$ $\forall x \in C$ IS THE **SUPREMUM OF f**

$$\sup_{x \in C} f(x) = \beta$$

- LARGEST $\alpha \in \mathbb{R}$ ST. $f(x) \ge \alpha$ $\forall x \in C$ IS THE **INFIMUM OF f**

$$\inf_{x \in C} f(x) = \alpha$$

(IF NOT FAMILIAR WITH sup & inf — SEE BOOK §.2.3)

• FOR (P) IF THE INFIMUM OF $f(x)$ OVER FEASIBILITY REGION F EXIST THEN

$$MP = \inf_{x \in F} \{ f(x) \}$$

x SATISFY $x \in C$

$g_i(x) \leq 0 \ \forall i$

• IF x^* IS A SOLUTION OF (P) THEN

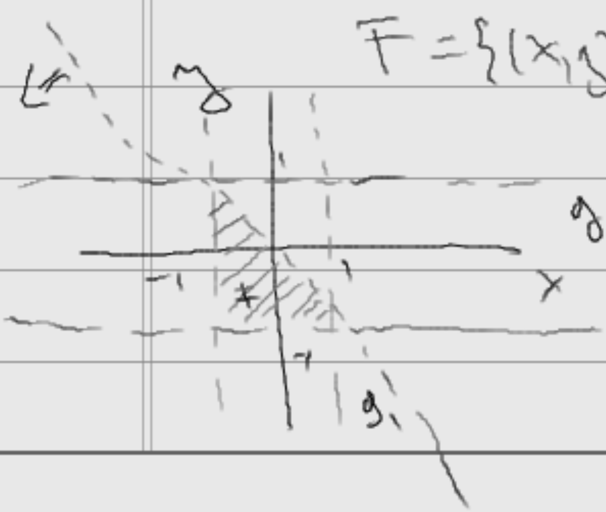
$$MP = f(x^*)$$

EXAMPLE:

(P)
$$\begin{cases} \text{MINIMIZE} & f(x,y) = x^4 + y^4 \\ \text{SUBJECT TO} & x^2 - 1 \leq 0 \quad g_1 \\ & y^2 - 1 \leq 0 \quad g_2 \\ & x + y - 1 \leq 0 \quad g_3 \\ \text{WHERE} & (x,y) \in \mathbb{R}^2 \end{cases}$$

FEASIBILITY REGION

$$F = \{(x,y) \in \mathbb{R}^2 : x + y \leq 0,\ |x| \leq 1,\ |y| \leq 1\}$$



$F \neq \emptyset \Rightarrow$ (P) IS CONSISTENT

F CONTAINS SLATER POINTS $(-0.5, -0.5) \Rightarrow$ (P) IS SUPERCONSISTENT

SINCE f, g_1, g_2, g_3, C ARE CONVEX, (P) IS A CONVEX PROGRAM, UNIQUE SOLUTION $(0,0)$, $MP = 0$ WHERE $f(0,0) = MP$

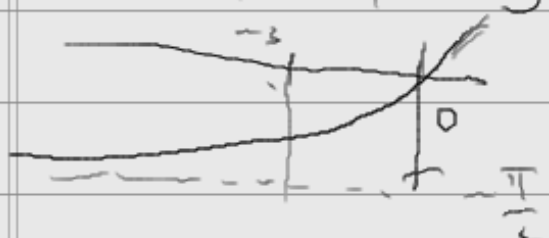
EXAMPLE: MP MAY EXIST, BUT NO SOLUTION

MINIMIZE $\arctan x$

S.T. $x + 3 \leq 0$

WHERE $x \leq 0$

$F = (-\infty, -3]$, $MP = \inf_{x \leq -3} \arctan(x) = -\frac{\pi}{2}$



BUT NO SOLUTION AS $\arctan(x) > -\frac{\pi}{2}$ $\forall x \in F$

DEF:

LINEAR PROGRAM IS A PROGRAM WHERE $f, g_1, \ldots g_m$ ARE LINEAR FUNCTIONS AND C IS $\mathbb{R}^n$ (OR $(\mathbb{R}^{+0})^n$)

HOW MUCH CHANGES IF SUBJECT TO CHANGES?

DEF

$z \in \mathbb{R}^m$

$$(P(z)) \begin{cases} \text{MINIMIZE } f(x) \quad \text{S.T.} \\ g_1(x) \leq z_1, \quad g_2(x) \leq z_2, \ldots g_m(x) \leq z_m \\ \text{WHERE } x \in C \subseteq \mathbb{R}^n, \; f, g_i \text{ CARE CONVEX} \end{cases}$$

(P(z)) IS A CONVEX PROGRAMM AS $g_i(x) - z_i \leq 0$ IS CONVEX

(P) IS (P(0))

INTERESTING IF z SMALL ... PERTURBATION

DEF

LET $F(z)$ BE THE FEASIBILITY REGION OF (P(z)). THEN

$$MP(z) = \inf_{x \in F(z)} \{f(x)\}$$

LET $h : z \to MP(z)$ (FOR $z \in \mathbb{R}^m$ ST $\mathcal{F}(z) \neq \emptyset$)

" IF NO $\inf_{x \in \mathcal{F}(z)} \{ f(x) \}$ EXISTS, THEN $MP(z) = -\infty$

$h : \mathbb{R}^m \to \mathbb{R} + \{-\infty\}$

THEOREM 46 (5.26)

MP(z) IS CONVEX AND ITS DOMAIN D IS CONVEX. IF THE PROGRAM P IS SUPER-CONSISTENT, THEN $0 \in D^\circ$.

$\square$

NOTES:

- $z_1, z_2 \in D$, $MP(z_1) = -\infty$ THEN

$$\forall z \in [z_1, z_2) : MP(z) = -\infty$$

$$z = \lambda z_1 + (1-\lambda) z_2 \quad \lambda \in (0, 1]$$

$$MP(z) \le \lambda MP(z_1) + (1-\lambda) z_2 = -\infty$$

$$= -\infty \ \forall \lambda > 0$$

- $\exists z \in D : MP(z) = -\infty$ THEN

$$MP(z') = -\infty \ \forall z' \in D^\circ$$

• $\exists z \in \overset{\circ}{D} : MP(z) \in \mathbb{R}$ THEN

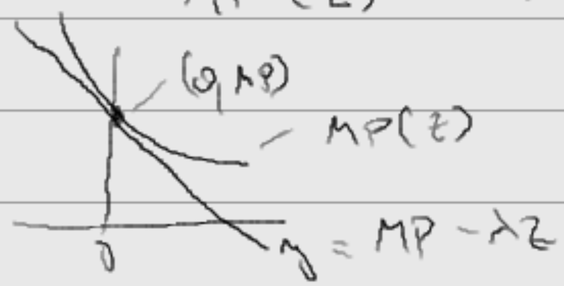
$$MP(z) \in \mathbb{R} \quad \forall z \in D$$

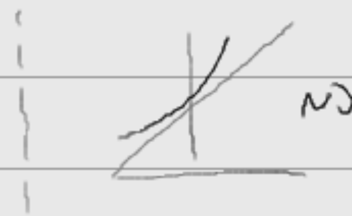
THEOREM 47 (5.2.8)

LET (P) BE SUPER CONSISTENT & CONVEX

LET $MP(0) \in \mathbb{R}$, THEN $\exists \lambda \geq 0 \in \mathbb{R}^m$

$$MP(z) \geq MP(0) - \lambda \cdot z \quad \forall z \in D$$





PROOF: ($z < 0$ STRICT CONSTR., $z > 0$ RELAXED)

SUPPORT THEOREM (45) $\Rightarrow \exists \lambda \in \mathbb{R}^m$

$$MP(z) \geq MP(0) - \lambda . z$$

COORDINATE

FOR CONTRADICTION LET $\lambda_i < 0$.

946 : $0 \in D^\circ \Rightarrow \exists r > 0$ ST $B(0, r) \subseteq D^\circ$

$z^i = (0, \dots \underset{\text{i-TH}}{\frac{r}{2}}, 0 \dots) \in B(0, r)$

$$MP(z^i) \geq MP(0) - \lambda . z^{(i)} = MP - \frac{r}{2}\lambda_i$$

$$MP(z^i) > MP$$

BUT $z \geq 0 \Rightarrow$ WEAKER CONSTRAINTS IN $(P(z)) \Rightarrow MP(z) \leq MP(0) = MP$

$\square$

<u>DEF:</u>

$\lambda \in \mathbb{R}^m$, $\lambda \geq 0$ IS SENSITIVITY VECTOR OF A CONVEX (P) IF

$$MP(z) \geq MP(0) - \lambda \cdot z \qquad \forall z \in D$$

NOTE: λ IS $\nabla MP(0)$ IF EXISTS

THM 47 – λ ALWAYS EXISTS FOR SUPERCONSISTENT (P)

EXAMPLES

$$(P)\begin{cases} \text{MINIMIZE} & e^{x+y} \\ \text{S.T.} & -x-y \leq 0 \end{cases}$$

$$(P(z))\begin{cases} \text{MINIMIZE} & e^{x+y} \\ \text{S.T.} & -x-y \leq z \qquad (x+y \geq -z) \end{cases}$$