• $\exists\, z \in D^\circ : MP(z) \in \mathbb{R}$ THEN

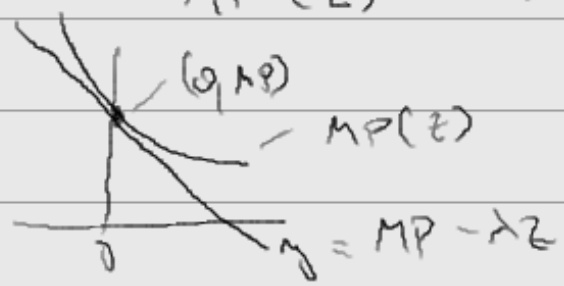
$$MP(z) \in \mathbb{R} \quad \forall z \in D$$

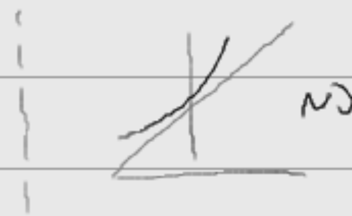
THEOREM 47 (5.2.8)

LET (P) BE SUPER CONSISTENT & CONVEX

LET $MP(0) \in \mathbb{R}$, THEN $\exists\, \lambda \geq 0 \in \mathbb{R}^m$

$$MP(z) \geq MP(0) - \lambda \cdot z \quad \forall z \in D$$





PROOF: ($z<0$ STRICT CONSTR., $z>0$ RELAXED)

SUPPORT THEOREM (45) $\Rightarrow \exists\, \lambda \in \mathbb{R}^m$

$$MP(z) \geq MP(0) - \lambda . z$$

COORDINATE

FOR CONTRADICTION LET $\lambda_i < 0$.

946 : $0 \in D^\circ \Rightarrow \exists\, r > 0$ ST $B(0, r) \subseteq D^\circ$

$z^i = (0, \ldots, \frac{r}{2}, 0 \ldots) \in B(0, r)$

i-TH

$$MP(z^i) \geq MP(0) - \lambda . z^{(i)} = MP - \frac{r}{2}\lambda_i$$

$$MP(z^i) > MP$$

BUT $z' \geq 0 \Rightarrow$ WEAKER CONSTRAINTS IN $(P(z')) \Rightarrow MP(z') \leq MP(0) = MP$

$\square$

DEF:

$\lambda \in \mathbb{R}^m$, $\lambda \geq 0$ IS SENSITIVITY VECTOR OF A CONVEX (P) IF

$$MP(z) \geq MP(0) - \lambda \cdot z \quad \forall z \in D$$

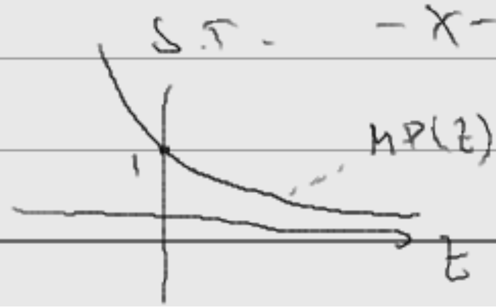
NOTE: λ IS $\nabla MP(0)$ IF EXISTS

THM 47 – λ ALWAYS EXISTS FOR SUPERCONSISTENT (P)

EXAMPLES

$$(P) \begin{cases} \text{MINIMIZE } e^{x+y} \\ \text{S.T. } -x-y \leq 0 \end{cases}$$

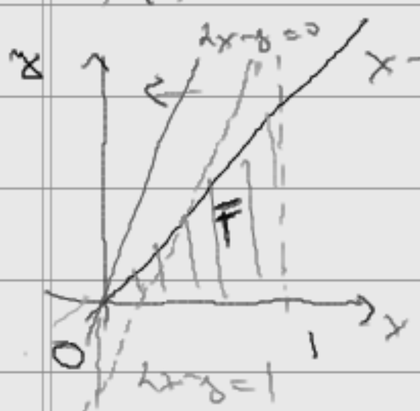
$$(P(z)) \begin{cases} \text{MINIMIZE } e^{x+y} \\ \text{S.T. } -x-y \leq z \end{cases} \quad (x+y \geq -z)$$



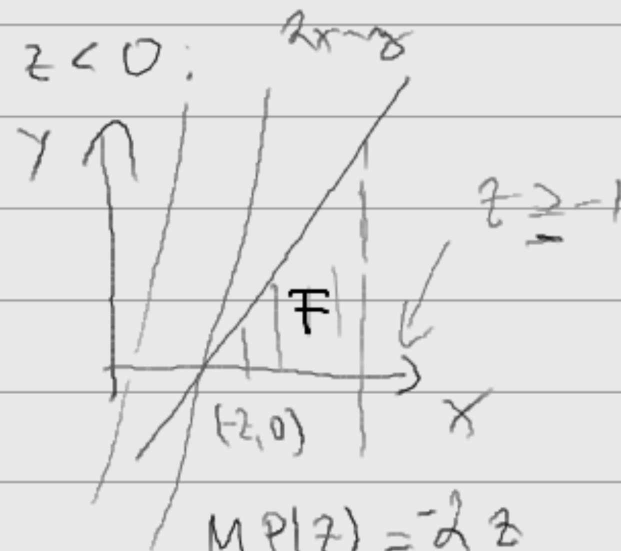
$$(P)\begin{cases} \text{MINIMIZE } 2x-y \\ \text{S.T. } -x+y \le 0 \\ \text{WHERE } y \ge 0 \\ \qquad 0 \le x \le 1 \end{cases}$$

$$(P(z))\begin{cases} \text{MINIMIZE } 2x-y \\ \text{S.T. } -x+y \le z \\ \qquad y\ge 0,\ 0\le x\le 1 \end{cases}$$

$MP = 0$

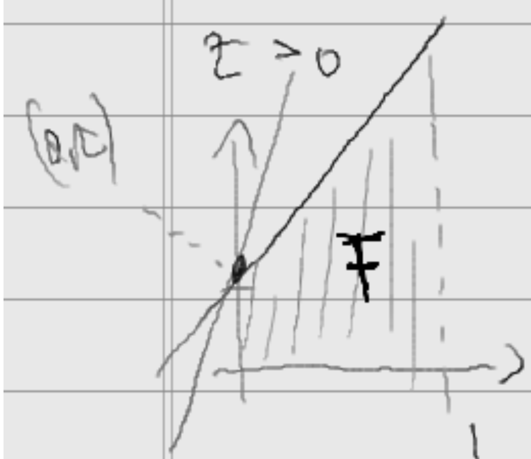


$z<0$:



$MP(z) = -2z$

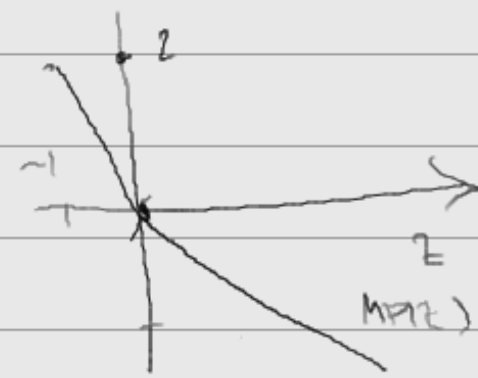
$z>0$



$-x+y=z$

$y=z$

$MP(z) = -z$



EX (NOT CONTINUOUS $MP(z)$)

$$(P)\quad\begin{cases}\text{MINIMIZE} & e^{-y}\\ \text{S.T.} & \sqrt{x^2+y^2}-x\le 0\end{cases}$$

$\because\ \sqrt{x^2+y^2}\ge x \Rightarrow y=0$

$MP(0)=e^{-0}=1 \leftarrow F=\{(x,0): x\ge 0\}$

$$(P(z))\quad\begin{cases}\text{MIN} & e^{-y}\\ \text{S.T.} & \sqrt{x^2+y^2}-x\le z \quad\Rightarrow \boxed{z\ge 0}\end{cases}$$

$$\sqrt{x^2+y^2}\le z+x$$

$$x^2+y^2\le z^2+2xz+x^2$$

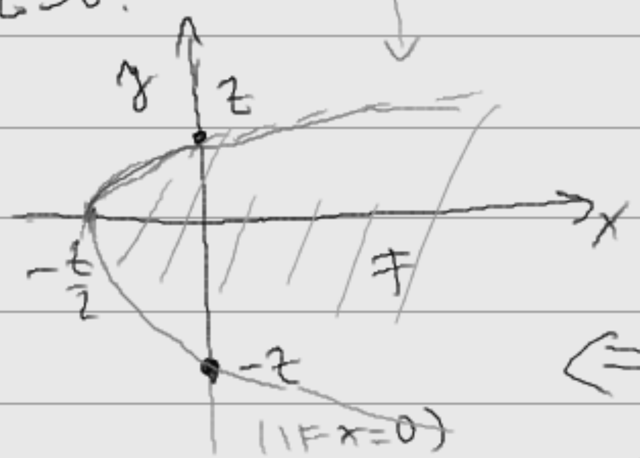
$$y^2\le z^2+2xz$$

$z>0$:

$y^2=z^2$

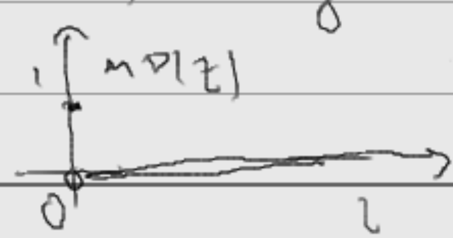
$-z\le 2x$

$x=\dfrac{y^2-z^2}{2z}$



$\Leftarrow$ y IS NOT BOUNDED

$MP(z)=\inf e^{-y}=0$



EXAMPLE FOR λ

$$(P)\begin{cases} \text{MINIMIZE } f(x) \\ \text{S.T. } g_1(x) \le 0,\ g_2(x) \le 0,\ g_3(x) \le 0 \end{cases}$$

LET SENSITIVITY VECTOR $\lambda = (100, 1, 0)$

THM 47: $z = (z_1, z_2, z_3)$

$$MP(z) \geq MP(0) - \lambda . z = MP - 100 z_1 - z_2$$

$MP((1,0,0)) \geq MP - 100$ ← IMPORTANT

$MP((0,1,0)) \geq MP - 1$

$MP((0,0,1)) \geq MP$ ← USELESS

THEOREM 48 (5.2.11)

LET

$$(P)\begin{cases} \text{MINIMIZE } f(x) \\ \text{ST. } g_1(x) \le 0, \dots g_m(x) \le 0 \\ \text{WHERE } x \in C \end{cases}$$

BE CONVEX AND $\lambda \in \mathbb{R}^m$ ITS SENSITIVITY VECTOR. THEN

$$MP = \inf_{x \in C} \left\{ f(x) + \sum_{i=1}^{m} \lambda_i g_i(x) \right\}$$

UNCONSTRAINED OPTIMIZATION JAHOO!

PROOF: [MP $\leq$ inf]

BY DEFINITION $MP(z) \geq MP - \lambda, z$

NOTE $\forall x \in C$, $z = (g_1(x), \ldots g_m(x))$ THEN

x IN DOMAIN OF MP(z).

$$MP(z) + \sum \lambda_i g_i(x) \geq MP$$

BY DEF (MP IS inf) $f(x) \geq MP(z)$

$$f(x) + \sum \lambda_i g_i(x) \geq MP \quad \Leftarrow \forall x \in C$$

$$\inf_{x \in C} \left\{ f(x) + \sum_{i=1}^{m} \lambda_i g_i(x) \right\} \geq MP$$

[MP $\geq$ inf]

$$\inf_{x \in C} \left\{ f(x) + \sum_{i=1}^{m} \lambda_i g_i(x) \right\} \leq \inf_{\substack{x \in C \\ g(x) \leq 0}} \left\{ f(x) + \sum_{i=1}^{m} \lambda_i g_i(x) \right\} \leq$$

$\lambda_i \geq 0$

$$\leq \inf_{x \in C} \{ f(x) \}$$

$\square$

(P) SUPER CONSISTENT $\overset{T47}{\Longrightarrow}$ $\exists$ SENSITIVITY VECTOR $\overset{T48}{\Longrightarrow}$

CAN BE TRANSLATED AS UNCONSTRAINED OPT.!

THEOREM 49 (5.2.13 - KARUSH-KUHN-TUCKER, SADDLE POINT FORM)

LET (P) BE A SUPERCONSISTENT CONVEX PROGRAM. THEN $x^* \in C$ IS A SOLUTION OF (P) IFF $\exists \lambda^* \in \mathbb{R}^m$ S.T.

(1) $\lambda^* \geq 0$

(2) $L(x^*, \lambda) \leq L(x^*, \lambda^*) \leq L(x, \lambda^*)$ $\forall x \in C$, $\forall \lambda \geq 0$ SADDLE POINT

(3) $\lambda_i^* g_i(x^*) = 0 \quad \forall i = 1, 2, \ldots m$

PROOF:

$(\Longrightarrow)$ x^* IS A SOLUTION OF (P) (1)

$x^* \in C$, $g(x^*) \leq 0$, $f(x^*) = MP$, T47 $\Rightarrow \exists \lambda^* \geq 0$

SENSITIVITY VECTOR, T48:

$$f(x^*) = \inf_{x \in C} L(x, \lambda^*).$$

NEXT

$$f(x^*) \geq f(x^*) + \sum \lambda_i^* g_i(x^*) = L(x^*, \lambda^*)$$

$\lambda_i \geq 0$ $\quad$ $g_i(x^*) \leq 0$

$\Downarrow$ (2) $\quad$ $\Downarrow$ (3)

$L(x^*, \lambda^*) \leq L(x, \lambda^*)$ $\quad$ $\lambda_i^* g_i(x^*) = 0$

MISSING $L(x^*, \lambda) \leq L(x^*, \lambda^*) \quad \forall \lambda \geq 0$

$L(x^*, \lambda^*) - L(x^*, \lambda) =$

$= f(x^*) + \sum \underbrace{\lambda_i^* g_i(x^*)}_{= 0} - f(x^*) - \sum \lambda_i g_i(x^*) =$

$= - \sum \underbrace{\lambda_i}_{\geq 0} \underbrace{g_i(x^*)}_{\leq 0} \geq 0$

$\Downarrow$

$L(x^*, \lambda^*) \geq L(x^*, \lambda) \quad \forall \lambda$

$(\Longleftarrow)$

LET $x^* \in C$ AND $\lambda^* \in \mathbb{R}^m$ SATISFY (1), (2) & (3)

DEF $\lambda^{(i)}$ ST $\lambda_j^{(i)} = \begin{cases} \lambda_j^* & \text{IF } j \neq i \\ \lambda_j^* + 1 & \text{IF } j = i \end{cases}$

(SHIFT i^{TH} COORDINATE) $\lambda^{(i)} \geq 0$

(2) $\Rightarrow 0 \geq L(x^*, \lambda^{(i)}) - L(x^*, \lambda^*) =$

$= f(x^*) + \sum \lambda_i^{(i)} g_i(x) - f(x^*) - \sum \lambda_i^* g_i(x) = g_i(x)$

$\Rightarrow x^*$ IS A FEASIBLE SOLUTION

$$(3) \Rightarrow f(x^*) = f(x^*) + \sum_{i=1}^{m} \underbrace{\lambda_i^* g_i(x^*)}_{0} = L(x^*, \lambda^*)$$

$$L(x^*, \lambda^*) \overset{(2)}{=} \inf\{L(x, \lambda^*) : x \in C\} \leq$$

$$\leq \inf\{L(x, \lambda^*) : x \in F\} =$$

$$= \inf\{f(x) + \sum \underbrace{\lambda_i}_{\geq 0} \underbrace{g_i(x)}_{\leq 0},\ x \in F\}$$

$$\leq \inf\{f(x) : x \in F\} = MP \qquad \square$$

NOTE: (3) CAN BE PROVED FROM (2)

DEF:

- (x^*, λ^*); $x^* \in C$ $\lambda^* \geq 0$; ST

  (2) $L(x^*, \lambda) \leq L(x^*, \lambda^*) \leq L(x, \lambda^*) \quad \forall x \in C, \lambda \geq 0$

  IS CALLED THE SADDLE POINT OF THE LAGRANGIAN
- CONDITION $\lambda_i^* g_i(x^*) = 0$, $i = 1, 2, \dots m$ IS COMPLEMENTARY SLACKNESS CONDITION.

REPHRASE THM 4.9 USING SLACKNESS & SADDLE POINT