DEFINITION

THE LAGRANGIAN $L(x, \lambda)$ OF A CONVEX PROGRAM

$$(P) \begin{cases} \text{MIN. } f(x) \\ \text{S.T. } g_1(x) \leq 0 \ldots g_m(x) \leq 0 \\ x \in C \end{cases}$$

IS FUNCTION DEFINED BY

$$L(x, \lambda) = f(x) + \sum_{i=1}^{m} \lambda_i g_i(x)$$

$x \in C, \lambda \geq 0$

$L(x, \lambda) : \mathbb{R}^{n+m} \to \mathbb{R}$

(P) SUPERCONSISTENT WITH $MP \in \mathbb{R} \Rightarrow$

$$MP = \inf_{x \in C} \{L(x, \lambda)\}$$

MAIN THEOREM OF THE WEEK

THEOREM 49 (5.2.13 - KARUSH-KUHN-TUCKER, SADDLE POINT FORM)

LET (P) BE A SUPER CONSISTENT CONVEX PROGRAM. THEN $x^* \in C$ IS A SOLUTION OF (P) IFF $\exists \lambda^* \in \mathbb{R}^m$ S.T.

(1) $\lambda^* \geq 0$

SADDLE POINT

(2) $L(x^*, \lambda) \leq L(x^*, \lambda^*) \leq L(x, \lambda^*)$ $\forall x \in C$, $\forall \lambda \geq 0$

(3) $\lambda_i^* g_i(x^*) = 0 \quad \forall i = 1, 2, \ldots, m$

PROOF:

($\Longrightarrow$) x^* IS A SOLUTION OF (P) (1)

$x^* \in C$, $g(x^*) \leq 0$, $f(x^*) = MP$, T47 $\Rightarrow \exists \lambda^* \geq 0$

SENSITIVITY VECTOR, T48:

$$f(x^*) = \inf_{x \in C} L(x, \lambda^*).$$

NEXT

$$f(x^*) \geq f(x^*) + \sum \lambda_i^* g_i(x^*) = L(x^*, \lambda^*)$$

$\lambda_i \geq 0$ $\qquad$ $g_i(x^*) \leq 0$

$\Downarrow$ (2) $\qquad$ $\Downarrow$ (3)

$L(x^*, \lambda^*) \leq L(x, \lambda^*)$ $\qquad$ $\lambda_i^* g_i(x^*) = 0$

MISSING $L(x^*,\lambda) \leq L(x^*,\lambda^*) \quad \forall \lambda \geq 0$

$$L(x^*,\lambda^*) - L(x^*,\lambda) =$$

$$= f(x^*) + \sum \underbrace{\lambda_i^* g_i(x^*)}_{=0} - f(x^*) - \sum \lambda_i g_i(x^*) =$$

$$= -\sum \underbrace{\lambda_i}_{\geq 0} \underbrace{g_i(x^*)}_{\leq 0} \geq 0$$

$$\Downarrow$$

$$L(x^*,\lambda^*) \geq L(x^*,\lambda) \quad \forall \lambda$$

$(\Longleftarrow)$

LET $x^* \in C$ AND $\lambda^* \in \mathbb{R}^m$ SATISFY (1), (2) & (3)

DEF $\lambda^{(i)}$ ST $\lambda_j^{(i)} = \begin{cases} \lambda_j^* & \text{IF } j \neq i \\ \lambda_j^* + 1 & \text{IF } j = i \end{cases}$

(SHIFT i^{TH} COORDINATE) $\lambda^{(i)} \geq 0$

(2) $\Rightarrow$ $0 \geq L(x^*,\lambda^{(i)}) - L(x^*,\lambda^*) =$

$$= f(x^*) + \sum \lambda_i^{(i)} g_i(x) - f(x^*) - \sum \lambda_i^* g_i(x) = g_i(x)$$

$\Rightarrow$ x^* IS A FEASIBLE SOLUTION

$$(3) \Rightarrow \quad f(x^*) = f(x^*) + \sum_{i=1}^{m} \underbrace{\lambda_i^* g_i(x^*)}_{=0} = L(x^*, \lambda^*)$$

$$L(x^*, \lambda^*) \overset{(2)}{=} \inf \{ L(x, \lambda^*) : x \in C \} \leq$$

$$\leq \inf \{ L(x, \lambda^*) : x \in F \} =$$

$$= \inf \{ f(x) + \sum \underbrace{\lambda_i}_{\geq 0} \underbrace{g_i(x)}_{\leq 0}, \ x \in F \}$$

$$\leq \inf \{ f(x) : x \in F \} = MP \qquad \square$$

NOTE: (3) CAN BE PROVED FROM (2)

DEF:

- (x^*, λ^*) ; $x^* \in C$ $\lambda^* \geq 0$; ST

(2) $$L(x^*, \lambda) \leq L(x^*, \lambda^*) \leq L(x, \lambda^*) \quad \forall x \in C, \lambda \geq 0$$

IS CALLED THE SADDLE POINT OF THE LAGRANGIAN

- CONDITION $\lambda_i^* g_i(x^*) = 0$, $i = 1, 2, \ldots m$ IS COMPLEMENTARY SLACKNESS CONDITION.

REPHRASE THM 49 USING SLACKNESS & SADDLE POINT

NOTE: IF (x^*, λ^*) IS A SADDLE POINT OF LAGR. OF ANY CONVEX PROGRAM (P) THEN

(1) MP IS FINITE & x^* IS SOLUTION

(2) COMPLEMENTARY SLACKNESS CONDITION

$$(\lambda_i^* g_i(x^*) = 0 \quad i = 1, 2, \dots m)$$

IS SATISFIED

THEOREM 50 (5.2.14 KARUSH-KUHN-TUCKER GRADIENT FORM)

LET (P) BE SUPERCONSISTENT CONVEX PROGRAM AND $f, g_1, g_2 \dots g_m$ HAVE CONTINUOUS FIRST PARTIAL DERIVATIVES ON C FROM (P). IF x^* IS FEASIBLE FOR (P) AND IS AN INTERIOR POINT OF C, THEN x^* IS A SOLUTION $\boxed{\text{IFF}}$ $\exists \lambda^* \in \mathbb{R}^m$ SUCH THAT

(1) $\lambda_i^* \geq 0$ FOR $i = 1, 2, \dots m$

(2) $\lambda_i^* g_i(x^*) = 0$ FOR $i = 1, 2, \dots m)$

(3) $\nabla f(x^*) + \sum_{i=1}^{m} \lambda_i^* \nabla g_i(x^*) = 0$

PROOF ($\Rightarrow$)

T 4.9 $\Rightarrow \exists \lambda^* \in \mathbb{R}^m$ SATISFYING (1) & (2)

WHERE (x^*, λ^*) IS A SADDLE POINT.

HENCE $\quad L(x^*, \lambda^*) \leq L(x, \lambda^*) \ \forall x \in C$

$\Rightarrow$ x^* IS A GLOBAL MINIMIZER OF

$$h(x) = L(x, \lambda^*) = f(x) + \sum \lambda_i g_i(x)$$

$h(x)$ HAS CONTINUOUS FIRST PARTIAL DERIVATIVES.

HENCE $\nabla h(x^*) = 0 \Rightarrow (3)$.

($\Leftarrow$) $\qquad \square$

x^*, λ^* SATISFY 1, 2, 3

$\forall x$ F FEASIBLE SOL:

$$f(x) \geq f(x) + \sum \underbrace{\lambda_i^*}_{\geq 0} \overbrace{g_i(x)}^{\leq 0}$$

[$\forall$ CONVEX: $f(x) \geq f(x^*) + \nabla f(x^*)^T \cdot (x - x^*)$]

$$f(x) \geq f(x^*) + \nabla f(x^*)(x - x^*) + \sum \lambda_i (g_i(x^*) + $$
$$+ \nabla g_i(x^*)(x - x^*))$$

$$\Rightarrow f(x) \geq [f(x^*) + \sum \lambda_i g_i(x^*)] + \overbrace{[\nabla f(x^*) + \sum \lambda_i^* \nabla g_i(x^*)]}^{=0}(x - x^*)$$

$$f(x) \geq f(x^0) + \underbrace{\sum \lambda_i g_i(x^0)}_{\geq 0} + 0$$

$$\geq f(x^0)$$

$\Rightarrow$ x^0 IS A MINIMIZER OF f.

$\square$

EXAMPLE OF USAGE OF THM 50

MINIMIZE $-x_1 + x_2$

S.T. $x_1^2 + x_2 - 2 \leq 0$

$-2x_1 - x_2 - 1 \leq 0$

– SUPERCONSISTENT & DIFFERENTIABLE

$\Rightarrow$ FIND (x_1^0, x_2^0) AND $(\lambda_1^0, \lambda_2^0)$ S.T.

(1) $\lambda_1^0 \geq 0$ $\lambda_2^0 \geq 0$

(2) $\lambda_i g_i(x) = 0$ $\lambda_1^0(x_1^2 + x_2 - 2) = 0$

$\lambda_2^0(-2x_1 - x_2 - 1) = 0$

$$\nabla f(x) + \lambda_1 \nabla g(x) + \lambda_2 \nabla g_2(x) = 0$$

x_1: $\quad -1 + \lambda_1^* 2x_1^* - \lambda_2^* 2 = 0$

x_2: $\quad 1 + \lambda_1^* - \lambda_2^* = 0$

$$\left(\begin{array}{l} \lambda_1^*(x_1^{*2} + x_2^* - 2) = 0 \\ \lambda_2^*(-2x_1^* - x_2^* - 1) = 0 \end{array}\right)$$

- $\lambda_2^* = 0 \Rightarrow \lambda_1^* = -1 \Rightarrow \lightning$
- $\lambda_1^* = 0$:

$$\left.\begin{array}{l} -1 - 2\lambda_2 = 0 \\ 1 - \lambda_2 = 0 \end{array}\right. \Rightarrow \lightning\lightning$$

- $\lambda_1 \neq 0 ; \lambda_2 \neq 0$

$$x_1^{*2} + x_2^* - 2 = 0$$

$$-2x_1^* - x_2^* - 1 = 0$$

$$x_1^{*2} - 2x_1^* - 3 = 0$$

$$x_1^* = \frac{2 \pm \sqrt{4+12}}{2} = \frac{2 \pm 4}{2} = \left\langle \begin{array}{l} 3 \\ -1 \end{array}\right.$$

$x^* = (3, -7)$ or $(-1, 1)$

$x^* = (3, -7) \quad \left(-1 + \lambda_1^* \, 2x_1^* - \lambda_2^* \, 2 = 0 \right)$

$\lambda_2^* = \lambda_1^* + 1 \quad \Leftarrow \quad 1 + \lambda_1^* - \lambda_2^* = 0$

$-1 + 6\lambda_1^* - 2\lambda_2^* = 0$

$-1 + 6\lambda_1^* - 2\lambda_1^* - 2 = 0$

$4\lambda_1^* = 3$

$\lambda_1^* = \frac{3}{4} \Rightarrow \lambda_2 = \frac{7}{4}$

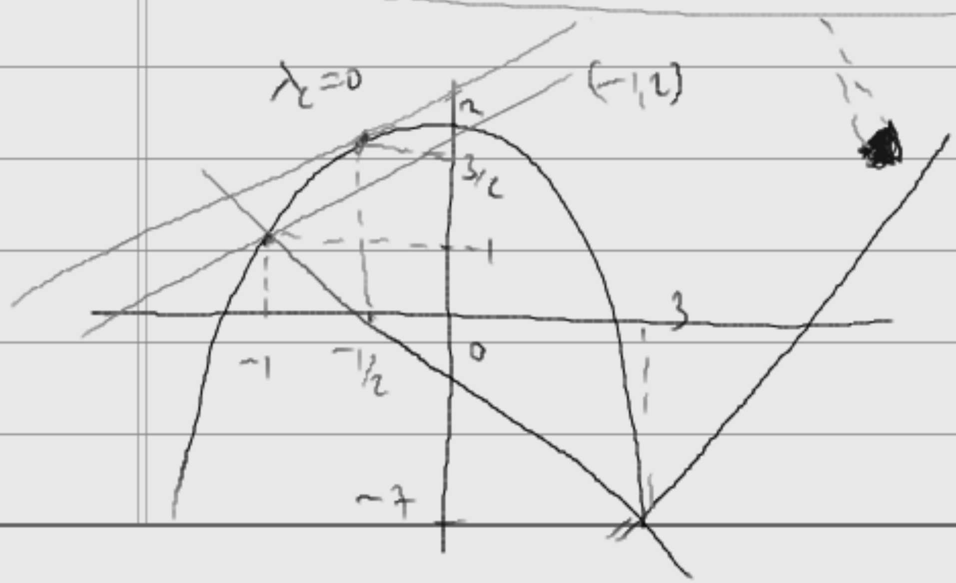
$x^* = (-1, 1)$

$-1 - 2\lambda_1 - 2\lambda_1 - 2 = 0$

$-5 = 4\lambda_1^*$

$\lambda_1^* = -\frac{3}{4}$???

$\boxed{\lambda^* = \left(\frac{3}{4}, \frac{7}{4}\right) \quad x^* = (3, -7) \quad f(x^*) = -10}$



THEOREM 51 (5.2.16)

LET (P) BE A CONVEX PROGRAM AND

(x^*, λ^*) SATISFY KKT CONDITIONS

THEN λ^* IS A SENSITIVITY VECTOR FOR (P)

(THAT IS $\quad MP(z) \geq MP - \lambda^* . z$)

$\square$

DEF:

$\lambda_1^*, \ldots, \lambda_n^*$ ARE KKT MULTIPLIERS OF (P)

$\Rightarrow$ EXAMPLE IS EQUIVALENT TO

MINIMIZE

$$Q(x_1, x_2) = -x_1 + x_2 + \frac{3}{4}(x^2 + x_2 - 2) + \frac{7}{4}(-2x_1 - x_2 - 1)$$