KKT AND DUAL IN LP

$$(LP)\begin{cases} \min & c^T x \\ \text{s.t.} & Ax \le b \\ & \forall i = 1 \dots n \end{cases} \qquad (DLP)\begin{cases} \max & b^T y \\ \text{s.t.} & A^T y = c \\ & y \le 0 \end{cases}$$

KKT (GRADIENT)

1) $\lambda \ge 0$

2) $Ax < b \Rightarrow \lambda = 0$; $\lambda \ge 0 \Rightarrow Ax = b$

(COMPLEMENTARY SLACKNESS)

3) $\nabla f(x^*) + \sum \lambda_i^{*T} \nabla g_i(x) = 0$

$\dots c + \lambda^T A = 0 \Rightarrow -A^T \lambda = c$

KKT (SADDLE POINT) - GIVES ALL (DLP)

1) $\lambda \ge 0$

2) COMPLEMENTARY SLACKNESS

3) $L(x^*, \lambda) \le L(x^*, \lambda^*) \le L(x, \lambda^*)$ $\qquad c^T = -\lambda^T A$

$L(x,\lambda) = c^T \cdot x + \lambda^T (Ax - b)$ $\qquad \Downarrow$

$\lambda \ge 0$, $Ax - b \le 0$ $\qquad c^T + \lambda^T A = 0$

$Ax - b = 0$ OR $y_i = 0$ (2)

$\min c^T x$ FROM 2 ; FOR λ: $\max \cdot c^T x - c^T x - \lambda^T b$

$$f(x) \geq f(x^0) + \underbrace{\sum \lambda_i g_i(x^0)}_{\geq 0} + 0$$

$$\geq f(x^0)$$

$\Rightarrow$ x^0 IS A MINIMIZER OF f.

$\square$

KKT - GRADIENT VERSION

EXAMPLE OF USAGE OF THM 50

MINIMIZE $-x_1 + x_2$

S.T. $x_1^2 + x_2 - 2 \leq 0$

$-2x_1 - x_2 - 1 \leq 0$

- SUPERCONSISTENT & DIFFERENTIABLE

$\Rightarrow$ FIND (x_1^*, x_2^*) AND $(\lambda_1^*, \lambda_2^*)$ S.T.

(1) $\lambda_1^* \geq 0$ $\lambda_2^* \geq 0$

(2) $\lambda_i g_i(x) = 0$ $\lambda_1^*(x_1^2 + x_2 - 2) = 0$

$\lambda_2^*(-2x_1 - x_2 - 1) = 0$

$$\nabla f(x) + \lambda_1 \nabla g_1(x) + \lambda_2 \nabla g_2(x) = 0$$

x_1: $\quad -1 + \lambda_1^* 2x_1^* - \lambda_2^* 2 = 0$

x_2: $\quad 1 + \lambda_1^* - \lambda_2^* = 0$

$$\begin{pmatrix} \lambda_1^*(x_1^{*2} + x_2^* - 2) = 0 \\ \lambda_2^*(-2x_1^* - x_2^* - 1) = 0 \end{pmatrix}$$

- $\lambda_2^* = 0 \Rightarrow \lambda_1^* = -1 \Rightarrow$ ↯
- $\lambda_1^* = 0$:

$$\left.\begin{array}{l} -1 - 2\lambda_2 = 0 \\ 1 - \lambda_2 = 0 \end{array}\right\} \Rightarrow ↯$$

- $\lambda_1 \neq 0$; $\lambda_2 \neq 0$

$$x_1^{*2} + x_2^* - 2 = 0$$

$$-2x_1^* - x_2^* - 1 = 0$$

$$x_1^{*2} - 2x_1^* - 3 = 0$$

$$x_1^* = \frac{2 \pm \sqrt{4+12}}{2} = \frac{2 \pm 4}{2} = \left\langle \begin{array}{l} 3 \\ -1 \end{array} \right.$$

$x^* = (3, -7)$ or $(-1, 1)$

$x^* = (3, -7) \quad \left(-1 + \lambda_1^* 2x_1^* - \lambda_2^* 2 = 0\right)$

$\lambda_2^* = \lambda_1^* + 1 \quad \Leftarrow \quad 1 + \lambda_1^* - \lambda_2^* = 0$

$-1 + 6\lambda_1^* - 2\lambda_2^* = 0$

$-1 + 6\lambda_1^* - 2\lambda_1^* - 2 = 0$

$4\lambda_1^* = 3$

$\lambda_1^* = \frac{3}{4} \implies \lambda_2 = \frac{7}{4}$

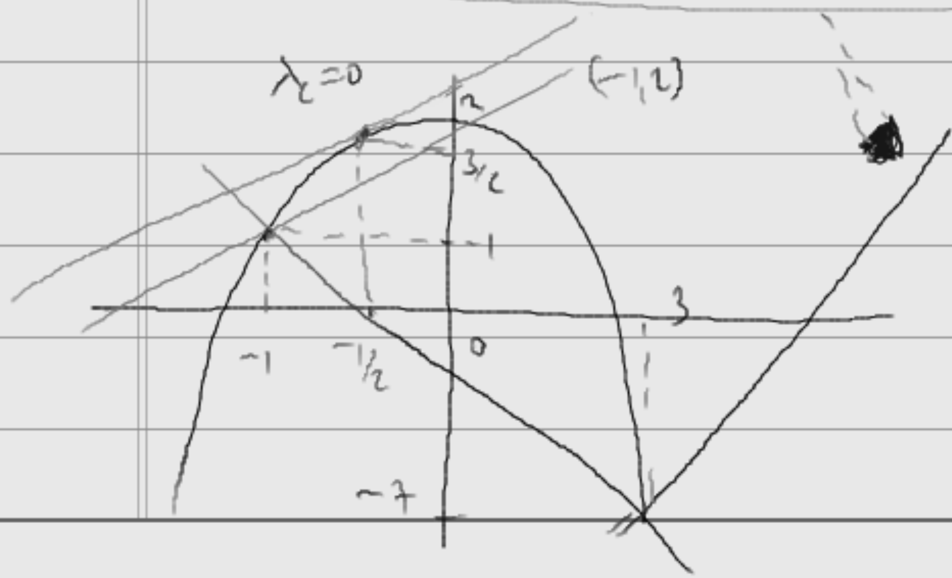
$x^* = (-1, 1)$

$-1 - 2\lambda_1 - 2\lambda_1 - 2 = 0$

$-5 = 4\lambda_1^*$

$\lambda_1^* = -\frac{3}{4}$ ↯↯↯

$\boxed{\lambda^* = \left(\frac{3}{4}, \frac{7}{4}\right) \quad x^* = (3, -7) \quad f(x^*) = -10}$



THEOREM 51 (5.2.16)

LET (P) BE A CONVEX PROGRAM AND

(x^*, λ^*) SATISFY KKT CONDITIONS (SADDLE)

THEN λ^* IS A SENSITIVITY VECTOR FOR (P)

(THAT IS $\forall z$: $MP(z) \geq MP - \lambda^{*T} z$)

[NOTE 49: SENSITIVITY VECTOR WORKS IN KKT

THIS IS THE OTHER WAY]

PROOF:

x^* OPT SOLUTION => MP FINITE

LET $z \in \operatorname{dom} MP(.)$ => $\forall x \in C$ S.T.

$\forall i \quad g_i(x) \leq c_i$

$MP = f(x^*) = f(x^*) + \sum \lambda_i^* g_i(x^*) = L(x^*, \lambda^*) \leq$

KKT

$\leq L(x, \lambda^*) = f(x) + \sum \lambda_i^* g_i(x)$ BY $\leq$

$\leq f(x) + \lambda^{*T} z$

$\Rightarrow MP \leq \inf\{f(x)\} + \lambda^{*T} z$

$\Rightarrow MP(z) \geq MP - \lambda^{*T} z$

$\square$

DEF:

$\lambda_1^*, \ldots, \lambda_n^*$ ARE KKT MULTIPLIERS OF (P)

$\Rightarrow$ EXAMPLE IS EQUIVALENT TO

MINIMIZE

$$Q(x_1, x_2) = -x_1 + x_2 + \frac{3}{4}(x^2 + x_2 - 2) + \frac{7}{4}(-2x_1 - x_2 - 1)$$

5.3. KKT & CONSTRAINED GEOMETRIC PROGRAMMING

THEOREM 24 (2.4.1) A-G INEQUALITY

LET $x_1 \ldots x_m \in \mathbb{R}^+$, $\delta_1 \ldots \delta_m \in (0,1)$ ST $\sum_{i=1}^{n} \delta_i = 1$ THEN

$$\prod_{i=1}^{n} x_i^{\delta_i} \leq \sum_{i=1}^{m} \delta_i x_i$$

WITH EQUALITY IFF $x_1 = x_2 = \ldots x_m$

THEOREM 52 (5.3.1, EXTENDED A-G INEQ).

LET $X_1, \ldots X_m \in \mathbb{R}^+$. LET $\delta_1 \ldots \delta_m$ BE ALL POSITIVE OR ALL ZERO.

IF $\lambda = \delta_1 + \ldots + \delta_m$ THEN

$$\left(\sum_{i=1}^{m} x_i \right)^{\lambda} \geq \lambda^{\lambda} \left(\prod_{i=1}^{m} \left(\frac{x_i}{\delta_i} \right)^{\delta_i} \right)$$

WHERE $0^0 = 1$ AND $(x_i/0)^0 = 1$

WITH EQUALITY IFF $\delta_1 = \delta_2 = \ldots = \delta_m = 0$

OR $\forall_i \; x_i = \frac{\delta_i}{\lambda} \left(\sum_{j=1}^{m} x_j \right)$

PROOF - USE (AG)

- $\forall_i \; \delta_i > 0$

$$\frac{\delta_1}{\lambda} + \frac{\delta_2}{\lambda} + \frac{\delta_3}{\lambda} + \dots + \frac{\delta_m}{\lambda} = 1$$

$$\sum_{i=1}^{m} x_i = \sum_{i=1}^{m} \left(\frac{\delta_i}{\lambda}\right)\left(\frac{\lambda x_i}{\delta_i}\right) \underset{(A-G)}{\geq} \prod_{i=1}^{m} \left(\frac{\lambda x_i}{\delta_i}\right)^{\left(\frac{\delta_i}{\lambda}\right)}$$

(A-G) WITH EQUALITY IFF

$$\frac{\lambda x_1}{\delta_1} = \frac{\lambda x_2}{\delta_2} = \dots = \frac{\lambda x_m}{\delta_m} = M$$

$$\left(\sum_{i=1}^{m} x_i\right)^{\lambda} \geq \prod_{i=1}^{m} \left(\frac{\lambda x_i}{\delta_i}\right)^{\delta_i} = \lambda^{\lambda} \prod_{i=1}^{m} \left(\frac{x_i}{\delta_i}\right)^{\delta_i}$$

$$\sum x_i = \sum \frac{M \delta_i}{\lambda} = \frac{M}{\lambda} \underbrace{\sum \delta_i}_{=\lambda} = M$$

$$\frac{\lambda x_i}{\delta_i} = \sum x_i \Rightarrow x_i = \frac{\delta_i}{\lambda}\left(\sum x_i\right)$$

- $\forall_i \; \delta_i = 0$ TRIVIAL $1 = 1$

$\square$