5.3. KKT & CONSTRAINED GEOMETRIC PROGRAMMING

THEOREM 24 (2.4.1) A-G INEQUALITY

LET $x_1 \dots x_m \in \mathbb{R}^+$, $\delta_1 \dots \delta_m \in (0,1)$ ST $\sum_{i=1}^{n} \delta_i = 1$ THEN

$$\prod_{i=1}^{n} x_i^{\delta_i} \leq \sum_{i=1}^{m} \delta_i x_i$$

WITH EQUALITY IFF $x_1 = x_2 = \dots x_m$

THEOREM 52 (5.3.1, EXTENDED A-G INEQ).

LET $X_1, \dots X_m \in \mathbb{R}^+$. LET $\delta_1 \dots \delta_m$ BE ALL POSITIVE OR ALL ZERO.

IF $\lambda = \delta_1 + \dots + \delta_m$ THEN

$$\left(\sum_{i=1}^{m} x_i\right)^{\lambda} \geq \lambda^{\lambda} \left(\prod_{i=1}^{m} \left(\frac{x_i}{\delta_i}\right)^{\delta_i} \right)$$

WHERE $0^0 = 1$ AND $(x_i/0)^0 = 1$

WITH EQUALITY IFF $\delta_1 = \delta_2 = \dots = \delta_m = 0$

OR $\forall_i \; x_i = \frac{\delta_i}{\lambda} \left(\sum_{j=1}^{m} x_j \right)$

PROOF — USE (AG)

- $\forall_i \; \delta_i > 0$

$$\frac{\delta_1}{\lambda} + \frac{\delta_2}{\lambda} + \frac{\delta_3}{\lambda} + \dots + \frac{\delta_m}{\lambda} = 1$$

$$\sum_{i=1}^{m} x_i = \sum_{i=1}^{m} \left(\frac{\delta_i}{\lambda}\right)\left(\frac{\lambda x_i}{\delta_i}\right) \underset{(A-G)}{\geq} \prod_{i=1}^{m} \left(\frac{\lambda x_i}{\delta_i}\right)^{\left(\frac{\delta_i}{\lambda}\right)}$$

(A-G) WITH EQUALITY IFF

$$\frac{\lambda x_1}{\delta_1} = \frac{\lambda x_2}{\delta_2} = \dots = \frac{\lambda x_m}{\delta_m} = M$$

$$\left(\sum_{i=1}^{m} x_i\right)^{\lambda} \geq \prod_{i=1}^{m} \left(\frac{\lambda x_i}{\delta_i}\right)^{\delta_i} = \lambda^{\lambda} \prod_{i=1}^{m} \left(\frac{x_i}{\delta_i}\right)^{\delta_i}$$

$$\sum x_i = \sum \frac{M \delta_i}{\lambda} = \frac{M}{\lambda} \underbrace{\sum \delta_i}_{=\lambda} = M$$

$$\frac{\lambda x_i}{\delta_i} = \sum x_i \Rightarrow x_i = \frac{\delta_i}{\lambda}\left(\sum x_i\right)$$

- $\forall_i \; \delta_i = 0$ TRIVIAL $1 = 1$

$\square$

DEF - GEOMETRIC PROGRAM

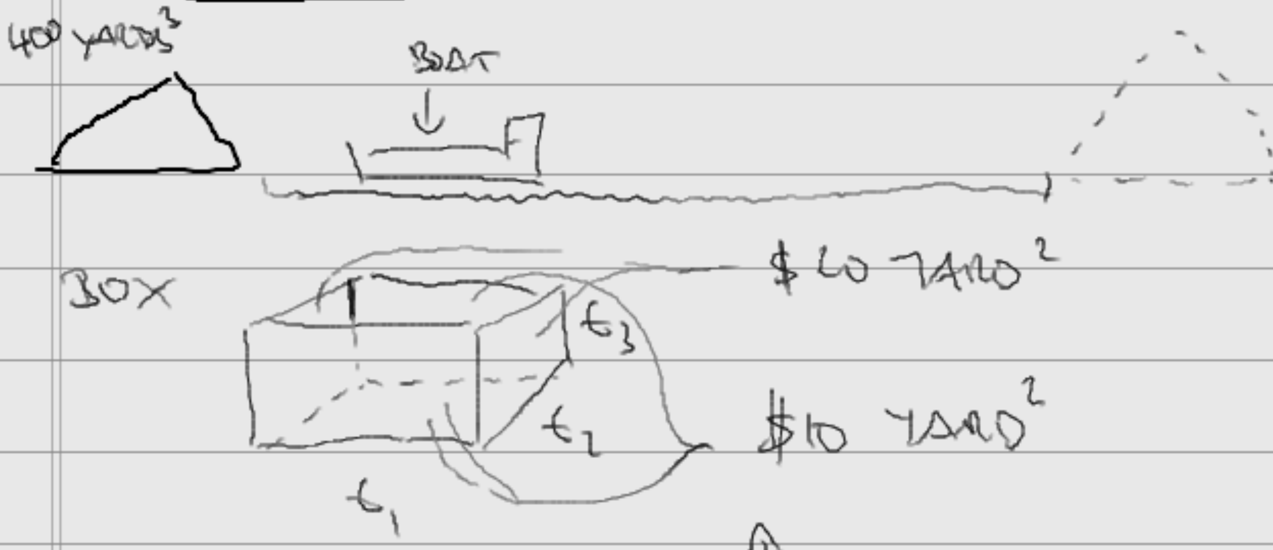
LET $g_0(t), g_1(t), \ldots g_k(t)$ BE POSYNOMIALS IN m REAL VARIABLES $t = (t_1, \ldots t_m)$. THEN THE PROGRAM

$$\text{MINIMIZE } g_0(t)$$
$$\text{S.T. } g_1(t) \le 1, \ldots g_k(t) \le 1$$
$$\text{WHERE } t_1 > 0, t_2 > 0, \ldots t_m > 0$$

IS CALLED CONSTRAINED GEOMETRIC PROGRAM.

RE DEF POSYNOMIAL $\sum c_i \prod t_j^{\alpha_{ij}}$

EXAMPLE



1 ROUND \$0.1

MINIMIZE COST

NO VALUE AFTER GRAVEL TRANSPORTED

MINIMIZE $0.1 \frac{400}{t_1 t_2 t_3} + 2.20\, t_2 t_3 + 2.10\, t_1 t_3 + 10\, t_1 t_2$

WHERE $t_1 > 0,\ t_2 > 0,\ t_3 > 0$

& NO CONSTRAINTS.

MINIMIZE USING (AG)

$$5\left(\frac{1}{5}\frac{20}{t_1 t_2 t_3} + \frac{1}{5}\frac{20}{t_1 t_2 t_3} + \frac{1}{5} 40 t_2 t_3 + \frac{1}{5} 20 t_1 t_3 + \frac{1}{5} 10 t_1 t_2\right) \geq$$

$$5\left(20.20.\ 40.\ 20.10\right)^{\frac{1}{5}} = 5 \cdot 20 = 100$$

$t_2 t_3 = \frac{1}{2}$ $\qquad t_1 = 2$

$t_1 t_3 = 1$ $\qquad t_3 = 1/2$

$t_1 t_2 = 2$ $\qquad t_2 = 1$

$t_1 t_2 t_3 = 1$

$\Rightarrow$ COST $= 40 + 20 + 20 + 20 = 100$

LETS CHANGE THE SETTING

WE HAVE 4 YARDS^2 OF THE SCRAP

MATERIAL FOR FREE BUT NOT MORE

MIN. $\frac{40}{t_1 t_2 t_3} + 40 t_2 t_3$

S.T $2t_1 t_3 + t_1 t_2 \le 4$

WHERE $t_1 > 0$ $t_2 > 0$ $t_3 > 0$ $(t = (t_1, t_2, t_3))$

$$g_0(t) = \frac{40}{t_1 t_2 t_3} + 40 t_2 t_3$$

$$g_1(t) = \frac{t_1 t_3}{2} + \frac{t_1 t_2}{4}$$

4'

$\circ$ IF $\lambda > 0$, $[g_1(t)]^\lambda \le 1$ (A) 4000

$$g_0(t) \ge g_0(t)\left(g_1(t)\right)^\lambda = \left(\frac{40}{t_1 t_2 t_3} + 40 t_2 t_3\right)(g_1(t))^\lambda$$

$$= \left(\delta_1\left(\frac{40}{\delta_1 t_1 t_2 t_3}\right) + \delta_2\left(\frac{40 t_2 t_3}{\delta_2}\right)\right)(g_1(t))^\lambda \ge$$

$\delta_1, \delta_2 > 0$, $\delta_1 + \delta_2 = 1$

$$\ge \left(\left(\frac{40}{\delta t_1 t_2 t_3}\right)^{\delta_1}\left(\frac{40 t_2 t_3}{\delta_2}\right)^{\delta_2}\right)\left(g_1(t)\right)^\lambda$$

$$= \left(\left(\frac{40}{\delta_1}\right)^{\delta_1} \cdot \left(\frac{40}{\delta_2}\right)^{\delta_2} (t_1)^{-\delta_1} t_2^{-\delta_1+\delta_2} t_3^{-\delta_1+\delta_2}\right)(g_1(t))^\lambda$$

$$(g_1(t))^{\lambda} = \left(\frac{t_1 t_3}{2} + \frac{t_1 t_2}{4}\right)^{\lambda}$$

$$\geq \lambda^{\lambda}\left(\left(\frac{t_1 t_3}{2\delta_3}\right)^{\delta_3}\left(\frac{t_1 t_2}{4\delta_4}\right)^{\delta_4}\right)$$

$$= \lambda^{\lambda}\left(\left(\frac{1}{2\delta_3}\right)^{\delta_3}\left(\frac{1}{4\delta_4}\right)^{\delta_4}\right) t_1^{(\delta_3+\delta_4)}\, t_2^{\delta_4}\, t_3^{\delta_3}$$

WHERE $\lambda = (\delta_3 + \delta_4)$

$$g_0(t) \geq \left(\left(\frac{40}{\delta_1}\right)^{\delta_1}\left(\frac{40}{\delta_2}\right)^{\delta_2}\left(\frac{1}{2\delta_3}\right)^{\delta_3}\left(\frac{1}{4\delta_4}\right)^{\delta_4}\right)\cdot(\delta_3+\delta_4)^{(\delta_3+\delta_4)}$$

$$\cdot\, t_1^{-\delta_1+\delta_3+\delta_4}\; t_2^{-\delta_1+\delta_2+\delta_4}\; t_3^{-\delta_1+\delta_2+\delta_3}$$

CANCELING $t_1\, t_2\, t_3$:

$$\begin{aligned}
\delta_1 + \delta_2 \qquad\qquad\qquad &= 1 \\
\delta_3 + \delta_4 &= \lambda \\
-\delta_1 \qquad + \delta_3 + \delta_4 &= 0 \\
-\delta_1 + \delta_2 \qquad + \delta_4 &= 0 \\
-\delta_1 + \delta_2 \qquad + \delta_3 &= 0
\end{aligned}$$

$\delta_1 > 0,\ \delta_2 > 0,\ \delta_3 \geq 0\ \ \delta_4 \geq 0$

SOLUTION IS

$$\delta_1 = \frac{2}{3} \quad \delta_2 = \frac{1}{3} \quad \delta_3 = \frac{1}{3} \quad \delta_4 = \frac{1}{3}$$

$$\Rightarrow g_0(t) \geq 60$$

WE WANT EQUALITIES IN BOTH (A-G) INEQ.

$$\frac{t_1^{\circ} t_3^{\circ}}{\frac{3}{3}} = \frac{t_1^{\circ} t_2^{h}}{\frac{4}{3}} = K$$

$$1 = \frac{t_1^{\circ} t_2^{\circ}}{2} + \frac{t_1^{\circ} t_2^{3}}{4} = \delta_3 K + \delta_4 K = \frac{2}{3} K$$

$$\Rightarrow K = \frac{3}{2}$$

$$\boxed{\begin{array}{l} t_1^{\circ} t_3^{b} = 1 \\ t_1^{\circ} t_2^{c} = 2 \end{array}} \qquad \in (A-G)$$

(A4) :

$$\frac{40}{\frac{2}{3} t_1 t_2 t_3} = \frac{40\, t_1^{+} t_2^{\circ}}{\frac{1}{3}} = t = 60$$

$$\frac{2}{3} z + \frac{1}{3} z = 60 \Rightarrow z = 60$$

$$\Rightarrow \quad \epsilon_1^* \epsilon_2^* \epsilon_3^* = 1$$

$$2 \epsilon_2^* \epsilon_3^* = 1$$

$$\boxed{\begin{aligned} \epsilon_1^* \epsilon_3^* &= 1 \\ \epsilon_1^* \epsilon_2^* &= 2 \end{aligned}} \leftarrow \text{FROM OLD}$$

log ALL

$$\log \epsilon_1^* + \log \epsilon_2^* + \log \epsilon_3^* = 0$$

$$\log \epsilon_2^* + \log \epsilon_3^* = -\log 2$$

$$\log \epsilon_1^* + \log \epsilon_3^* = 0$$

$$\log \epsilon_1^* + \log \epsilon_2^* = \log 2$$

$$\log \epsilon_1^* = \log 2 = \epsilon_1^* = 2$$

$$\log \epsilon_2^* = 0 \Rightarrow \epsilon_2^* = 1$$

$$\log \epsilon_3 = \log 2^{-1} \Rightarrow \epsilon_3^* = -\frac{1}{2}$$

DONE

$\rightarrow$ SHOW WHY A-G NOT GOOD

FORMALIZE PREVIOUS EXAMPLE

$$M_j(t) = C_j\, t_1^{\alpha_{j1}}\, t_2^{\alpha_{j2}}\, t_3^{\alpha_{j3}} \cdots t_m^{\alpha_{jm}} \qquad (c_j > 0)$$

CONSTRAINED (GP)

(GP)

MINIMIZE $g_0(t) = M_1(t) + \dots + M_{m_0}(t)$

SUBJECT TO

$g_1(t) = M_{m_0+1}(t) + \dots + M_{m_1}(t) \leq 1$

$\vdots$

$g_K(t) = M_{m_{K-1}+1}(t) + \dots + M_{m_K}(t) \leq 1$

WHERE $t > 0$

SET $m_K = p$

AG:

$$g_0(t) \geq g_0(t)\, g_1(t)^{\lambda_1}\, g_2(t)^{\lambda_2} \cdots g_K^{\lambda_K}(t) \geq$$

$$\geq \left(\prod_{j=1} \frac{c_j}{\delta_j}\right)^{\delta_j} \prod_{i=1}^{K} \lambda_i^{\lambda_i} \left(t^0 \cdot t^0 \cdots t^0\right)$$

(DGP)

MAXIMIZE $v(\delta) = \left(\prod_{j=1}^{p} \frac{c_j}{\delta_j}\right)^{\delta_j} \prod_{i=1}^{K} \lambda_i^{\lambda_i}$

S.T. $\delta_1 + \dots + \delta_{m_0} = 1$ (JUST AG)

$\alpha_{11}\delta_1 + \alpha_{21}\delta_2 + \dots + \alpha_{p1}\delta_p = 0$

$\vdots$

$\alpha_{1m}\delta_1 + \dots + \alpha_{pm}\delta_p = 0$

WHERE $\delta_i > 0$ FOR $1 \le i \le m_0$

$\forall k$ $\delta_i > 0$ FOR $m_{k-1}+1 \le i \le m_k$ OR

$\delta_i = 0$ FOR $m_{k-1} \le i \le m_k$

$\lambda_i = \delta_{m_{i-1}+1} + \dots + \delta_{m_i}$

DUAL OBJECTIVE FUNCTION, DUAL CONSTRAINTS,

DGP IS LINEAR .. WORKS = 1 SOLUTION

EXAMPLE:

$$\min \; t_1^{2} t_2 t_3^{-1}$$

$$2 t_1 t_2^{-2} + \frac{1}{2} t_3^{-1} \le 1 \qquad \lambda_1$$

$$\frac{1}{4} t_1^{-2} + \frac{1}{2} t_2^{2} t_3^{2} \le 1 \qquad \lambda_2$$

δ_1, δ_2, δ_3, δ_4, δ_5

WHERE $t_i > 0$

$$\max \left(\frac{1}{\delta_1}\right)^{\delta_1} \left(\frac{2}{\delta_2}\right)^{\delta_2} \left(\frac{1}{2\delta_3}\right)^{\delta_3} \left(\frac{1}{4\delta_4}\right)^{\delta_4} \left(\frac{1}{2\delta_5}\right)^{\delta_5}$$

$$\lambda_1 \quad \cdot (\delta_2+\delta_3)^{\delta_2+\delta_3} \cdot (\delta_4+\delta_5)^{\delta_4+\delta_5} \quad \lambda_2$$

SUBJECT TO

OBJ.	δ_1					$= 1$
t_1	$2\delta_1$	$+\delta_2$		$-2\delta_4$		$= 0$
t_2	δ_1	$-2\delta_2$			$+2\delta_5$	$= 0$
t_3	$-\delta_1$		$-\delta_3$		$+2\delta_5$	$= 0$

$\delta_1 > 0, \quad \delta_2 \delta_2 > 0$ OR $\delta_2 = \delta_3 = 0$

$\delta_4 \delta_5 > 0$ OR $\delta_4 = \delta_5 = 0$

$\delta_1 = 1$	
$\delta_2 = r$	$r \geq 0$
$\delta_4 = 1 - \frac{r}{2}$	$r \leq 2$
$\delta_5 = -\frac{1}{2} + r$	$r \geq \frac{1}{2}$
$\delta_3 = -2 + 2r$	$r \geq 1$

USING KKT FOR GP: (KKT FOR CONVEX)

MIN $f(x)$

S.T. $g_1(x) \leq 0, \; g_2(x) \leq 0, \ldots \; g_m(x) \leq 0$

f, g CONVEX, $x \in C$ CONVEX

POSYNOMIAL IS NOT CONVEX: e.g. ... $t^{\frac{1}{2}}$

(iii) $t > 0 \Rightarrow \exists!\, x : e^x = t$

$t_j = e^{x_j}$

$$g(t) = \sum_{i=1}^{m} c_i t_1^{\alpha_{i1}} t_2^{\alpha_{i2}} \dots t_m^{\alpha_{im}}$$

$$h(x) = \sum_{i=1}^{m} c_i e^{\sum_{j=1}^{m} \alpha_{ij} x_j}$$

(ii) $h(x)$ IS CONVEX

$$(GP) \begin{cases} \text{MIN } g_0(t) \\ \text{ST } g_1(t) \le 1 \quad g_2(t) \le 1 \dots g_k(t) \le 1 \\ \text{WHERE } t > 0 \end{cases}$$

ASSOCIATED CONVEX PROGRAM IS

$t_i \Rightarrow e^{x_i}$

$$(GP^*) \begin{cases} \text{MIN } h_0(x) \\ h_1(x) - 1 \le 0 \quad \dots \quad h_k(x) - 1 \le 0 \\ x \in \mathbb{R}^m \end{cases}$$

(ii) e^x STRICTLY INCREASING $\Rightarrow$

$t^* = (t_1^*, t_2^* \dots t_m^*)$ IS SOLUTION OF (GP)

IFF $x^* = (x_1^* \dots x_m^*)$ IS SOLUTION OF (GP^*)