FORMALIZE PREVIOUS EXAMPLE

$$u_j(t) = c_j\, t_1^{\alpha_{j1}}\, t_2^{\alpha_{j2}}\, t_3^{\alpha_{j3}} \cdots t_m^{\alpha_{jm}} \qquad (c_j > 0)$$

CONSTRAINED (GP)

$$(GP)\quad \begin{cases} \text{MINIMIZE } g_0(t) = u_1(t) + \dots + u_{n_0}(t) \\ \text{SUBJECT TO} \\ g_1(t) = u_{n_0+1}(t) + \dots + u_{n_1}(t) \le 1 \\ \vdots \\ g_k(t) = u_{n_{k-1}+1}(t) + \dots + u_{n_k}(t) \le 1 \\ \text{WHERE } t > 0 \end{cases}$$

SET $n_k = p$

AG:

$$g_0(t) \ge g_0(t)\, g_1(t)^{\lambda_1}\, g_2(t)^{\lambda_2} \cdots g_k^{\lambda_k}(t) \ge$$
$$\ge \left(\prod_{j=1} \frac{c_j}{\delta_j}\right)^{\delta_j} \prod_{i=1}^{k} \lambda_i^{\lambda_i} \left(t^0\, t^0 \cdots t^0\right)$$

$$(DGP)\quad \begin{cases} \text{MAXIMIZE } v(\delta) = \left(\prod_{j=1}^{p} \frac{c_j}{\delta_j}\right)^{\delta_j} \prod_{i=1}^{k} \lambda_i^{\lambda_i} \\ \text{S.T.} \quad \delta_1 + \dots + \delta_{n_0} = 1 \quad (\text{JUST AG}) \\ \alpha_{11}\delta_1 + \alpha_{21}\delta_2 + \dots + \alpha_{p1}\delta_p = 0 \\ \vdots \\ \alpha_{1m}\delta_1 + \dots + \alpha_{pm}\delta_p = 0 \end{cases}$$

WHERE $\delta_i > 0$ FOR $1 \le i \le m_0$

$\forall k$ $\quad \delta_i > 0$ FOR $m_{k-1}+1 \le i \le m_k$ OR

$\delta_i = 0$ FOR $m_{k-1} \le i \le m_k$

$\lambda_i = \delta_{m_{i-1}+1} + \dots + \delta_{m_i}$

DUAL OBJECTIVE FUNCTION, DUAL CONSTRAINTS,

DGP IS LINEAR .. WCKY = 1 SOLUTION

EXAMPLE:

δ_1

MIN $\quad t_1^{2}\, t_2\, t_3^{-1}$ $\qquad \delta_3$

$\delta_1 \quad 2t_1 t_2^{-2} + \frac{1}{2} t_3^{-1} \le 1 \qquad \lambda_1$

δ_2

$\frac{1}{4} t_1^{-2} + \frac{1}{2} t_2^{2} t_3^{2} \le 1 \qquad \lambda_2$

$\delta_4 \qquad \delta_5$

WHERE $t_i > 0$

$$\text{MAX} \quad \left(\frac{1}{\delta_1}\right)^{\delta_1} \left(\frac{2}{\delta_2}\right)^{\delta_2} \left(\frac{1}{2\delta_3}\right)^{\delta_3} \left(\frac{1}{4\delta_4}\right)^{\delta_4} \left(\frac{1}{2\delta_5}\right)^{\delta_5}$$

$$\lambda_1 \quad \cdot (\delta_2+\delta_3)^{\delta_2+\delta_3} \cdot (\delta_4+\delta_5)^{\delta_4+\delta_5} \quad \lambda_2$$

SUBJECT TO

OBJ.	δ_1					$=1$
t_1	$2\delta_1$	$+\delta_2$		$-2\delta_4$		$=0$
t_2	δ_1	$-2\delta_2$			$+2\delta_5$	$=0$
t_3	$-\delta_1$		$-\delta_3$		$+2\delta_5$	$=0$

$\delta_1 > 0$, $\delta_2\delta_3 > 0$ or $\delta_2 = \delta_3 = 0$

$\delta_4\delta_5 > 0$ or $\delta_4 = \delta_5 = 0$

$\delta_1 = 1$

$\delta_2 = r \qquad r \geq 0$

$\delta_4 = 1 - \frac{r}{2} \qquad r \leq 2$

$\delta_5 = -\frac{1}{2} + r \qquad r \geq \frac{1}{2}$

$\delta_3 = -2 + 2r \qquad r \geq 1$

USING KKT FOR GP: (KKT FOR CONVEX)

MIN $f(x)$

S.T. $g_1(x) \leq 0$, $g_2(x) \leq 0$, ... $g_n(x) \leq 0$

f, g CONVEX, $x \in C$ CONVEX

POSYNOMIAL IS NOT CONVEX: e.g. ... $t^{\frac{1}{2}}$

(iii) $t > 0 \Rightarrow \exists!\, x : e^x = t$

$t_j = e^{x_j}$

$$g(t) = \sum_{i=1}^{m} c_i t_1^{\alpha_{i1}} t_2^{\alpha_{i2}} \dots t_m^{\alpha_{im}}$$

$$h(x) = \sum_{i=1}^{m} c_i e^{\sum_{j=1}^{m} \alpha_{ij} x_j}$$

(ii) $h(x)$ IS CONVEX

$$(GP)\begin{cases} \text{MIN } g_0(t) \\ \text{ST } \quad g_1(t) \le 1 \quad g_2(t) \le 1 \dots g_k(t) \le 1 \\ \text{WHERE } t > 0 \end{cases}$$

ASSOCIATED CONVEX PROGRAM IS

$t_i \Rightarrow e^{x_i}$

$$(GP^*)\begin{cases} \text{MIN } h_0(x) \\ h_1(x) - 1 \le 0 \quad \dots \quad h_k(x) - 1 \le 0 \\ x \in \mathbb{R}^m \end{cases}$$

(ii) e^x STRICTLY INCREASING $\Rightarrow$

$t^* = (t_1^*, t_2^* \dots t_m^*)$ IS SOLUTION OF (GP)

IFF $x^* = (x_1^* \dots x_m^*)$ IS SOLUTION OF (GP^*)

AND $t_i^* = e^{x_i^*} \ \forall i$

$\boxed{\text{EVENING START}}$

THEOREM 53 (5.3.5 DUALITY OF GP)

1) LET t BE A FEASIBLE VECTOR FOR (GP) AND δ BE A FEASIBLE VECTOR FOR CORRESPONDING DUAL PROGRAM (DGP) THEN

$$g_0(t) \geq v(\delta) \quad \text{(PRIMAL-DUAL INEQUALITY)}$$

2) LET (GP) BE SUPERCONSISTENT AND t^* BE ITS SOLUTION. THEN CORRESPONDING (DGP) IS CONSISTENT AND HAS SOLUTION δ^* ST

$$g_0(t^*) = v(\delta^*)$$

AND

$$\delta_i^* = \begin{cases} \dfrac{u_i(t^*)}{g_0(t^*)} & i = 1 \dots n_0 \quad ! \\ \lambda_j(t^*)\, u_i(t^*) & i = m_{j-1}+1 \dots m_j \quad j = 1 \dots k \end{cases}$$

CONSEQUENCES:

- (GP) SUPERCONSISTENT & (DGP) INCONSISTENT
  $\Rightarrow$ NO SOLUTION
- EXISTS CERTIFICATE THAT t^* IS A SOLUTION
  (δ^* .. $g_0(t^*) = v(\delta^*)$)
- ALLOWS US TO SOLVE (DGP) INSTEAD OF (GP)
  (MAYBE DGP EASIER!)
- BOUND ERROR OF APPROXIMATE SOLUTION
  t AND δ S.T.

$$g_0(t) = v(\delta) + \varepsilon$$

MAX GAP (ERROR) OF SOLUTION

  ↳ APPROX. ALGORITHMS
- DUAL MIGHT HAVE A GOOD MEANING!

PROOF:

1) FROM (AG) INEQUALITY
2) IDEA.. CHANGE TO CONVEX PROGRAM
   APPLY KKT, MOVE KKT BACK $\Rightarrow$
GET EQUATIONS FOR t_i .. $\mathcal{L}_{\lambda_i \delta} = 0 \Rightarrow \delta^*$ FEASIBLE

CONSIDER $(GP)^{\circ}$ $(t_i = e^{x_i})$

SOLUTION t^* OF (GP) $\Rightarrow$ $\exists$ SOLUTION x^* OF $(GP)^{\circ}$

$(x_i = \ln t_i)$

(GP) SUPER CONSISTENT $\Rightarrow$ $(GP)^{\circ}$ SUPER CONSISTENT.

$\because$ FUNCTIONS IN (GP°) HAS CONT. FIRST PARTIAL DERIVATIVES.

$\Rightarrow$ KKT-GRADIENT FORM APPLIES (PAJO - 5.2.14)

$\Rightarrow \exists \lambda^* = (\lambda_1^*, \dots, \lambda_k^*)$ S.T.:

a) $\lambda_i^* \geq 0$ FOR $1 \leq i \leq k$

b) $\lambda_i^* (h_i(x^*) - 1) = 0$ FOR $1 \leq i \leq k$ (COMPL. SLACKNESS)

c) $\dfrac{\partial h_0}{\partial x_j}(x^*) + \sum_{i=1}^{k} \lambda_i^* \dfrac{\partial h_i}{\partial x_j}(x^*) = 0$ FOR $1 \leq j \leq m$

$\hookrightarrow$ WHAT IS THIS IN g ??

$$\frac{\partial h_i}{\partial x_j} = \frac{\partial h_i}{\partial t_j} \cdot \frac{dt_j}{dx_j} = \frac{\partial g_i}{\partial t_j} e^{x_j}$$

$e^{x_j} > 0$ $\therefore$ DIVIDE

$$(c) \quad \frac{\partial g_0}{\partial t_j}(t^*) + \sum_{i=1}^{k} \lambda_i^* \frac{\partial g_i}{\partial t_j}(t^*) = 0 \quad j = 1 \dots m$$

MULTIPLY BY t_j^*

$$(C'') \quad t_j^* \frac{\partial g_0}{\partial t_j}(t^*) + \sum_{i=1}^{k} \lambda_i^* t_j^* \frac{\partial g_i}{\partial t_j}(t^*) = 0$$

$\forall j = 1 \dots m$

RECALL: $g_i(t) = \sum M_q(t)$

$$M_q = C_q t_1^{\alpha_{q1}} t_2^{\alpha_{q2}} \dots t_m^{\alpha_{qm}}$$

$\rightarrow$ HOW THE DERIVATIVE LOOKS LIKE

(C'') IS $\forall j$ EQUIVALENT TO

$$\sum_{q=1}^{m_0} \alpha_{qj} M_q(t^*) + \sum_{r=1}^{k} \sum_{q=m_{r-1}+1}^{m_r} \lambda_r^* \alpha_{qj} M_q(t^*)$$

$$= 0$$

DIVIDE BY $g_0(t^*)$:

$$(d) \quad \sum_{q=1}^{m_0} \alpha_{qj} \left(\frac{M_q(t^*)}{g_0(t^*)} \right) + \sum_{r=1}^{k} \sum_{q=m_{r-1}+1}^{m_r} \alpha_{qj} \left(\frac{\lambda_r^* M_q(t^*)}{g_0(t^*)} \right) = 0$$

$$\delta_q^* = \begin{cases} \dfrac{M_q(t^*)}{g_0(t^*)} & q = 1, 2, \dots m_0 \\ \dfrac{\lambda_r^* M_q(t^*)}{g_0(t^*)} & q = m_{r-1}+1, \dots m_r \quad r = 1 \dots k \end{cases}$$

IS δ_q^* GOOD ??? FOR (DGP)

$$\delta_q^* > 0 \qquad q = 1 \dots m_0$$

- $\delta_q^* > 0$ FOR $q = m_{n-1}+1 \dots m_n$ IFF $\lambda_n^* > 0$

  → SO EITHER ALL ZERO OR ALL POSITIVE

- (d) SHOW THAT EXPONENT FOR $t_j = 0$

- (AS) INEQ-CONSTRAINT

$$\sum_{q=1}^{m_0} \delta_q^* = \sum_{q=1}^{m_0} \frac{u_q(t^*)}{g_0(t^*)} = \frac{g(t^*)}{g(t^*)} = 1$$

⇒ $\delta^* = (\delta_1^*, \dots, \delta_p^*)$ IS FEASIBLE VECTOR FOR (DGP).

IS δ^* IN THE FORM WE PROMISED ??

NOTE λ_n^* .. KKT MULTIPLIERS

$\lambda_n^*(\delta^*)$ .. FROM (DGP)

HOW $\lambda_n^*(\delta^*)$ LOOK ??

$$\lambda_n(\delta^*) = \sum_{q=m_{n-1}+1}^{m_n} \delta_q^* = \sum_{q=m_{n-1}+1}^{m_n} \lambda_n^* \frac{u_q(t^*)}{g_0(t^*)} = \lambda_n^* \frac{g_n(t^*)}{g_0(t^*)}$$

IS δ_q^* WHAT WE PROMISED ?? KKT ⇒

(b) $\lambda_n^* \left(g_n(t^*) - 1 \right) = 0 \qquad n = 1 \dots K$

SO $\lambda_r^\delta g_r(t^\delta) = \lambda_r^\delta$ $\lambda_r(\delta)$ $1 \le r \le K$

$$\delta_q^\delta = \frac{\lambda_r^\delta u_q(t^\delta)}{g_0(t^\delta)} = \frac{\lambda_r^\delta g_r(t^\delta) u_q(t^\delta)}{g_0(t^\delta)} = \lambda_r(\delta)\, u_q(t^\delta)$$

$\Downarrow$

THIS FORM WE PROVED!

NOW IS δ_q^δ OPTIMAL ??

1) $\Rightarrow$ $g_0(t^\delta) \ge v(\delta^\delta)$ (PRIMAL-DUAL-INEQ)

IS δ_q^δ FORCING EQUALITY OF $(AG)_s$?

$$g_0(t^\delta) = g_0(t^\delta) \cdot \left(g_1(t^\delta)\right)^{\lambda_1(\delta^\delta)} \cdot \ldots \cdot \left(g_K(t^\delta)\right)^{\lambda_K(\delta)} \overset{(A_s)}{\ge}$$

- $(\mathcal{L}^i) \Rightarrow$ $g_i(t^\delta) = 1$ OR $\lambda_i = 0 \Rightarrow \lambda_i(\delta^\delta) = 0$
  $\forall i = 1 \ldots k$

- δ_q^δ FOR $1 \le q \le n_0$ FORCE EQUALITY FOR (A_S)
  AT $g_0(t^\delta)$

MUST BE SAME ALL SAME

$$g_0(t^\delta) = \sum_{q=1}^{n_0} u_q = \sum_{q=1}^{n_0} \delta_q \left(\frac{u_q}{\delta_q}\right) = \sum_{q=1}^{n_0} \delta_q \left(\frac{u_q}{\frac{g_0(t^\delta)}{u_q}}\right)$$

$\square$

NOTE:

KKT λ_n^* .. SENSITIVITY OF SOLUTION

$$\lambda_n(\delta^*) = \frac{\lambda_n^*}{g_0(t^*)}$$

SAME

$\lambda_n(\delta^*)$ ALSO PROVIDES „SENSITIVITY" TO n^{TH} CONSTRAINT.

$\boxed{5.4}$ DUAL CONVEX PROGRAMS

DUALITY THEOREM FOR CONVEX PROGRAMS.

RECALL

$$P \begin{cases} \text{MIN. } f(x) \\ \text{S.T. } g_1(x) \le 0 \\ \quad \vdots \qquad\qquad x \in C \\ \quad g_m(x) \le 0 \\ \text{WHERE } f, g_1 \ldots g_m \text{ ARE CONVEX AND} \\ \quad x \in C,\ C \text{ CONVEX} \end{cases}$$

$$\underset{(-\infty)}{MP} = \inf \{ f(x) : x \in F^{\,x \in C,\ g_i(x) \le 0} \}$$