$\boxed{5.4}$ DUAL CONVEX PROGRAMS

DUALITY THEOREM FOR CONVEX PROGRAMS.

RECALL

$$P \begin{cases} \text{MIN.} & f(x) \\ \text{S.T.} & g_1(x) \le 0 \\ & \vdots \qquad\qquad x \in C \\ & g_m(x) \le 0 \\ & \text{WHERE } f, g_1 \ldots g_m \text{ ARE CONVEX AND} \\ & x \in C, \ C \text{ CONVEX} \end{cases}$$

$$\underset{(-\infty)}{MP} = \inf \{ f(x) : x \in F \overset{x \in C,\ g_i(x) \le 0}{} \}$$

(GEOMETRIC -- FOR KKT, MORE AGAIN FOR KKT)

KKT - SADDLEPOINT VERSION

x^* SOLUTION $\iff \exists \lambda^*$ ST

1) $\lambda^* \geq 0$

2) $L(x^*, \lambda) \leq L(x^*, \lambda^*) \leq L(x, \lambda^*) \quad \forall x \in C, \forall \lambda \geq 0$

3) $\lambda_i^* g_i(x^*) = 0 \quad \forall i = 1, 2 \dots m$

2): MAX λ .. AND FITS $L(x^*, \lambda^*)$ .. LOOKS LIKE DUALITY

$$\inf_{x \in C} L(x, \lambda) \leq L(x^*, \lambda) \leq L(x^*, \lambda^*)$$

$$\Rightarrow \sup_{\lambda \geq 0} \left\{ \inf_{x \in C} L(x, \lambda) \right\} \leq L(x^*, \lambda^*)$$

$$L(x^*, \lambda^*) \leq \inf_{x \in C} \left(L(x, \lambda^*) \right) \leq \sup_{\lambda \geq 0} \left\{ \inf_{x \in C} L(x, \lambda) \right\}$$

HENCE $\quad L(x^*, \lambda^*) = \sup_{\lambda \geq 0} \left\{ \inf_{x \in C} L(x, \lambda) \right\}$

RECALL

$$MP = f(x^0) = f(x^0) + \sum_{i=1}^{m} \lambda_i^0 \underbrace{g_i(x^0)}_{=0 \text{ BY } 3)} = L(x^0, \lambda^0)$$

$$\Downarrow$$

$$MP = \sup_{\lambda \geq 0} \left\{ \inf_{x \in C} L(x, \lambda) \right\}$$

DEF: DUAL PROGRAM:

$$(DP) \begin{cases} \text{MAXIMIZE } h(\lambda) = \inf_{x \in C} L(x, \lambda) \\ \\ \text{WHERE } \lambda \geq 0 \end{cases}$$

λ IS FEASIBLE IF $\inf_{x \in C} L(x, \lambda) \neq -\infty$

HIDDEN CONDITIONS.

$$MD = \sup_{\lambda \geq 0} \left\{ \inf_{x \in C} L(x, \lambda) \right\}$$

$\overset{?}{=}$ $MD = MP$ (IF SOLUTIONS EXIST)

SOLVING (P)

1) WRITE (DP)

2) COMPUTE λ^* OF (DP)

3) MINIMIZE $L(x, \lambda^*)$ ON C

LINEAR PROGRAMMING DUALITY

$$(LP) \begin{cases} \min\ b^T x & b \in \mathbb{R}^m \\ \text{ST. } Ax \geq c & A \in \mathbb{R}^{m \times n} \\ \text{WHERE } x \geq 0 & c \in \mathbb{R}^n \end{cases}$$

AS CONVEX PROGRAM

$$(LP) \begin{cases} \min\ b^T x \\ \text{ST. } c_i - a^{(i)^T} x \leq 0 \end{cases} \qquad (P) \begin{cases} f(x) \\ g_i(x) \leq 0 \end{cases}$$

$a^{(i)}$ ... i-TH ROW.

LAGRANGIAN - BECAUSE OF THE DUAL

$$L(x, \lambda) = b^T x + \sum_{i=1}^{m} \lambda_i (c_i - a^{(i)^T} x) =$$

$$= \left(b - \sum_{i=1}^{m} \lambda_i a^{(i)}\right)^T x + \sum_{i=1}^{m} \lambda_i c_i =$$

$$\lambda_1 \square\, \overset{\lambda_2}{\square} + \lambda_3 \square$$

$$= \left(b - A^T \lambda\right)^T x + \lambda^T c$$

DUAL: $\max \inf_{x \in C} L(x, \lambda)$

FIND CONDITIONS

WHEN $\inf_{x \in C} L(x, \lambda) > -\infty$?

- SUPPOSE $(b - A^T x)_i < 0$

(& SHOW THAT INF $= -\infty$)

DEFINE $x^{(t)}$ FOR $t < 0$ $\quad x^{(t)} \geq 0$

$$x_j^{(t)} = \begin{cases} t\,(b - (A^T\lambda))_i & j = i \\ 0 & j \neq i \end{cases}$$

$$L(x^{(t)}, \lambda) = t\,\underbrace{\left((b - A^T\lambda)_i\right)^2}_{\text{CONST} > 0} + \underbrace{\lambda c}_{\text{CONST}}$$

$$\lim_{t \to -\infty} L(x^{(t)}, \lambda) = -\infty$$

HENCE $b - A^T\lambda \geq 0$ IS IT ENOUGH?

$$L(x,\lambda) = \underbrace{(b - A^T\lambda)^T}_{\text{ALL COORDINATES NONNEGATIVE}} x + \underbrace{\lambda^T c}_{\text{CONST}} \geq \lambda^T c$$

SO: $\mathcal{F} = \{ \lambda \in \mathbb{R}^m : \lambda \geq 0, \ A^T\lambda \leq 0 \}$

$$h(\lambda) = \inf_{\lambda \geq 0} L(x,\lambda) = (b - A^T\lambda)^T x + \lambda^T c \geq \lambda^T c$$

HENCE THE RIGHT x IS $x = 0$.

DUAL:

$$(DLP) \begin{cases} \text{MAXIMIZE } h(\lambda) = \lambda \cdot c \\ \text{ST. } A^T\lambda \leq b \qquad \lambda \geq 0 \end{cases}$$

THEOREM 54 (OUR METHOD GIVES ONLY SUPER CONSISTENCY)

(LP) HAS SOLUTION x^* IFF (DLP) HAS SOLUTION λ^* AND $f(x^*) = h(\lambda^*)$. $\square$

A LITTLE DISAPPOINTING DUALITY THM:

THEOREM 55 (5.4.6 – WEAK DUALITY OF CP)

LET $f, g_1, \dots, g_m$ CONVEX FUNCTIONS WITH CONTINUOUS FIRST PARTIAL DERIVATIVES DEFINED ON CONVEX $C \subseteq \mathbb{R}^n$. IF y IS FEASIBLE FOR (P) AND λ IS FEASIBLE FOR (DP) THEN

$$f(y) \geq h(\lambda) = \inf_{x \in C} L(x, \lambda)$$

MOREOVER, (P) AND (DP) ARE CONSISTENT AND FINITE AND $MP \geq MD$

PROOF

y, λ FEASIBLE $\Rightarrow$

$$f(y) \geq f(y) + \sum \lambda_i g_i(y) = L(y, \lambda)$$

$$\Rightarrow f(y) \geq \inf_{x \in C} L(x, \lambda) = h(\lambda)$$

$$\Downarrow$$

$$f(y) \geq \sup_{\lambda} h(\lambda) = MD$$

&

$$MP = \inf_{x \in F} \{ f(x) \} \geq h(\lambda)$$

$$\Rightarrow MP \geq MD$$

$\square$

(EQUALITY IN DUAL IS NOT ALWAYS POSSIBLE)

COROLLARY:

IF x IS FEASIBLE FOR (P), λ IS FEASIBLE FOR (DP), AND $f(x) = h(\lambda)$ THEN x IS A SOLUTION OF (P) AND λ IS A SOLUTION OF (DP).

PROOF: $f(x) \geq MP \geq MD \geq h(\lambda) \Rightarrow$ OPT. SOL. $\square$

DEF:

IF $MP > MD$ THEN (P) HAS A **DUALITY GAP**.

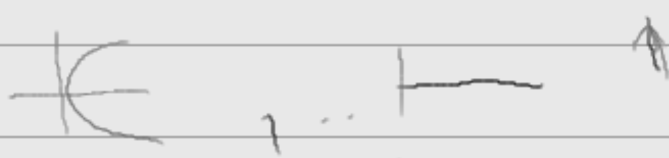
((P) SUPER CONSISTENT $\Rightarrow$ NO DUALITY GAP)

EXAMPLE - DUALITY GAP (WAS BEFORE)

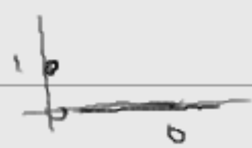
MINIMIZE $f(x, y) = e^{-y}$

S.T. $g(x, y) = \sqrt{x^2 + y^2} - x \leq 0$

WHERE $(x, y) \in \mathbb{R}^2$



$y = 0 \Rightarrow f(x, y) = 1$



MP(?) $\quad MP = 1$

$$(DP)\begin{cases} \text{MAXIMIZE } h(\lambda) = \inf\limits_{(x,y)\in\mathbb{R}^2} \{ e^{-y} + \lambda g(x,y) \} \\ \\ \text{S.T. } \lambda \geq 0 \end{cases}$$

$$MD = \sup \left[\inf_{(x,y)} \left\{ e^{-y} + \lambda \left[\sqrt{x^2+y^2} - x \right] \right\} \right]$$

$$\lim_{x\to\infty} \sqrt{x^2+y^2} - x = 0 \quad \forall \text{ FIXED } y$$

$$\lim_{y\to+\infty} \lim_{\lambda\to\infty} \left\{ e^{-y} + \lambda \left[\sqrt{x^2+y^2} - x \right] \right\} =$$

$$= \lim_{y\to+\infty} \left(e^{-y} \right) = 0 \quad \Rightarrow 0 \geq MD$$

$$\text{ALSO } MD \geq 0 \quad \Rightarrow \quad 0 = MD$$

$$MP = 1 > 0 = MD$$