6. PENALTY METHODS

$$(P)\quad \begin{cases} \min f(x) \\ g_1(x) \le 0, \ldots\ g_m(x) \le 0 \\ x \in \mathbb{R}^n \end{cases}$$

IDEA: PUT PENALTY FOR VIOLATED CONSTRAINTS TO OBJECTIVE FUNCTION

$$(\bar{P})\quad \min F(x), \quad x \in \mathbb{R}^n \ldots \qquad \text{OUR GOAL}$$

- LARGE VIOLATION $\Rightarrow$ LARGE INCREASE OF F
- ADJUST THE PENALTY

HOPE SOLUTION x_F^* IS CLOSE TO x^* ?

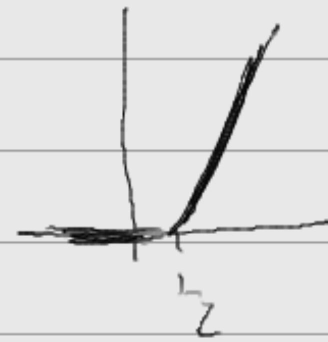
FOR $g_i(x)$ DEFINE $g_i^+(x) = \begin{cases} 0 & \text{IF } g(x) \le 0 \\ g(x) & \text{IF } g(x) > 0 \end{cases}$

$\rightarrow$ DOES BIG VIOLATION $\Rightarrow$ BIG PENALTY

EXAMPLES:

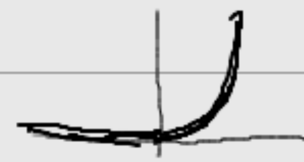
a) $g(x) = 2x - 1 \leq 0 \quad x \in \mathbb{R}$

$$g^+(x) = \begin{cases} 2x-1 & x > \frac{1}{2} \\ 0 & x \leq \frac{1}{2} \end{cases}$$



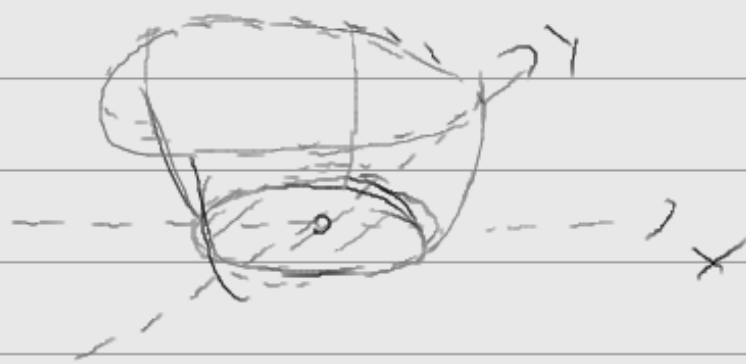
b) $g(x) = x^5$

$$g^+(x) = \begin{cases} x^5 & x > 0 \\ 0 & x = 0 \end{cases}$$



c) $g(x,y) = x^2 + y^2 - 1$

$$g^+(x,y) = \begin{cases} 0 & x^2 + y^2 \leq 1 \\ x^2+y^2-1 & \text{otherwise} \end{cases}$$



(P') $$F(x) = f(x) + \sum_{i=1}^{m} g_i^+(x)$$

DOES NOT CONTROL "SERIOUSNESS" OF PENALTY

(P') $$F_k(x) = f(x) + k \sum_{i=1}^{m} g_i^+(x)$$

CONTROL PENALTY - SOLVE FOR $k \to \infty$

WISH $\lim_{k \to \infty} x^*_k = x^*$

DEF:

ABSOLUTE VALUE PENALTY FUNCTION:

$\sum_{g_i \text{ is violated}} |g_i(x)|$ ← OUR CASE

k .. PENALTY PARAMETER

☹ g HAS DERIVATIVES ⇏ g^+ HAS DERIVATIVES.
(SEE EXAMPLES)

⇒ $F_k(x)$ LACK DERIVATIVES.

FIX BY (x^2) ← SEE X5

EXAMPLES

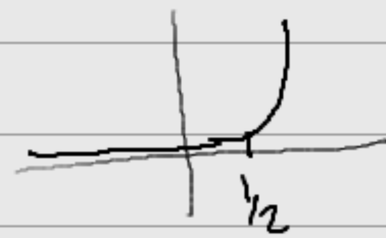
a) $g(x) = 2x - 1 \leq 0 \quad x \in \mathbb{R}$

$$g^+(x) = \begin{cases} 2x-1 & x > \frac{1}{2} \\ 0 & x \leq \frac{1}{2} \end{cases}$$

NO DERIVATIVE AT $x = 1/2$

$$h(x) = [g^+(x)]^2 = \begin{cases} (2x-1)^2 & x > \frac{1}{2} \\ 0 & x \leq \frac{1}{2} \end{cases}$$

now $h'(\frac{1}{2}) = 0$



b) $g(x,y) = x^2 + y^2 - 1$

$$g^+(x,y) = \begin{cases} 0 & x^2+y^2 \leq 1 \\ x^2+y^2-1 & \text{otherwise} \end{cases}$$

$h(x,y) = [g^+(x,y)]^2$

$\forall a,b$ ST. $a^2+b^2=1$:

$$\lim_{\substack{x \to a \\ y \to b}} \frac{\partial h}{\partial x}(x,y) = 0 = \lim_{\substack{x \to a \\ y \to b}} \frac{\partial h}{\partial y}(x,y)$$

LEMMA (6.1.3)

LET $g(x)$ HAS FIRST PARTIAL DERIVATIVES ON $\mathbb{R}^n$. THEN SO DOES $h(x) = [g^+(x)]^2$. MOREOVER,

$$\frac{\partial h}{\partial x_i}(x) = 2g^+(x) \cdot \frac{\partial g}{\partial x_i}(x)$$

$$\forall x \in \mathbb{R}^n \qquad i = 1 \dots n$$

$\square$

DEF:

$$P_k(x) = f(x) + k \sum_{i=1}^{m} [g_i^+(x)]^2$$

↳ COURANT - BELTRAMI PENALTY FUNCTION

PENALTY FUNCTION METHOD:

$$(P) \begin{cases} \text{MIN} & f(x) \\ \text{ST.} & g_1(x) \leq 0, \dots g_m(x) \leq 0 \quad x \in \mathbb{R}^n \end{cases}$$

1) $\forall k \in \mathbb{N}$: LET x_k^* BE GLOBAL MINIMIZER OF $P_k(x)$

2) SHOW THAT (SUB) SEQUENCE OF $\{x_k^*\}$ CONVERGES TO x^* — SOLUTION OF (P)

3) WHEN IT CONVERGES ?? — SEE LATER

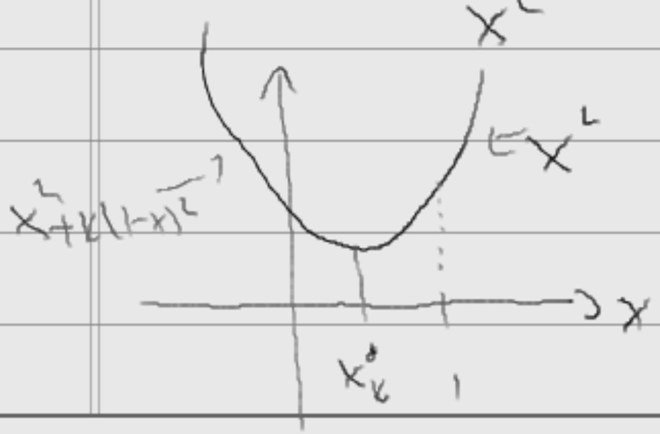
EXAMPLE:

$$(P) \begin{cases} \min \ f(x) = x^2 \\ \text{S.T.} \ \ g(x) = 1 - x \le 0 \end{cases} \qquad x \in \mathbb{R}$$

$1 \le x \Rightarrow x^* = 1 \quad f^P = 1 \leftarrow$ SIMPLE

$$P_k(x) = x^2 + k\,[(1-x)^+]^2$$

$$= \begin{cases} x^2 + k(1-x)^2 & x \le 1 \\ x^2 & x > 1 \end{cases}$$



FINDING MINIMIZER:

$P_k(x)$ IS DIFF. ERR.

$x > 1$ ... $P_k(x)$ INCREASING SO MINIMUM IS

$x \leq 1$.

$$0 = P_k'(x) = 2x + 2k(1-x)\cdot(-1) =$$
$$= (2+2k)x - 2k$$

$$x_k^* = \frac{2k}{2+2k} = \frac{k}{1+k}$$

SEQUENCE $\{x_k^*\}$ SATISFY:

1) $x_k^* < 1 \Rightarrow$ NONE OF x_k^* IS FEASIBLE FOR (P)

2) $\lim_{k \to \infty} x_k^* = 1 = x^*$ ← CONVERGING TO x^*

3) $P_k(x_k^*) = \left(\frac{k}{k+1}\right)^2 + k\left(1 - \frac{k}{k+1}\right)^2$

$$= \left(\frac{k}{k+1}\right)^2 + k\left(\frac{1}{k+1}\right)^2 =$$

$$= \frac{k^2+k}{(k+1)^2} = \frac{k}{k+1} < 1 = MP$$

SO

$$P_k(x_k^*) \leq MP \quad \forall k \text{ AND}$$

$$\lim_{k \to \infty} P_k(x_k^*) = MP.$$

1, 2, 3, ARE GENERALLY VALID

THEOREM 56 (6.2.3)

LET $f(x), g_1(x), \dots g_m(x)$ ARE CONTINUOUS ON $\mathbb{R}^n$ AND $f(x)$ IS BOUNDED FROM BELOW ($\exists c \in \mathbb{R}$ ST $f(x) \geq c \quad \forall x \in \mathbb{R}^n$). IF x^* IS A SOLUTION OF

$$(P) \begin{cases} \min f(x) \\ \text{S.T. } g_1(x) \leq 0 \dots g_m(x) \leq 0 \end{cases}$$

AND IF $\forall k \in \mathbb{N} \; \exists x_k \in \mathbb{R}^n$ ST.

$$P_k(x_k) = \min_{x \in \mathbb{R}^n} P_k(x)$$

THEN

1) $P_k(x_k) \leq P_{k+1}(x_{k+1}) \leq f(x^*) = MP \quad \forall k$

2) $\lim_{k \to \infty} \sum_{i=1}^{m} \left(g_i^+(x_k)\right)^2 = 0.$

HENCE IF $\{x_{k_p}\}$ IS A CONVERGING SUBSEQUENCE AND $\lim_{p \to \infty} x_{k_p} = x^{\infty}$ THEN x^{∞} IS A SOLUTION

OF (P).

NOTE:

- $\sum (g_i^+(x))^2$ COULD BE REPLACED BY $\sum g_i^+(x)$

- $P_k(x)$ COULD BE REPLACED BY $Q_k(x)$

$$Q_k(x) = f(x) + k \sum_{i=1}^{m} G_i(x)$$

WHERE

1) $G_i(x)$ IS CONTINUOUS IF $g_i(x)$ IS CONTINUOUS
2) $G_i(x) \geq 0 \quad \forall x \in \mathbb{R}^n$
3) $G_i(x) = 0$ IFF x IS FEASIBLE FOR (P)

G_i ... GENERALIZED PENALTY FUNCTION

COROLLARY

LET $f, g_1, \dots g_m$ ARE CONTINUOUS FUNCTIONS

AND

$$P = \begin{cases} \min.\ f(x) \\ \text{ST. } g_1(x) \leq 0, \dots g_m(x) \leq 0 \\ x \in \mathbb{R}^n \end{cases}$$

HAS SOLUTION x^*.

IF f IS COERCIVE AND IF

$$P_k(x) = f(x) + k \sum_{i=1}^{m} [g_i^+(x)]^2$$

THEN

1) $\forall k\ \exists x_k \in \mathbb{R}^n$ ST

$$P_k(x_k) = \min_{x \in \mathbb{R}^n} P_k(x)$$

2) SEQUENCE $\{x_k\}$ IS BOUNDED AND ALL CONVERGENT SUBSEQUENCES CONVERGE TO A SOLUTION OF (P)

PROOF:

MAYBE SKIP:

COERCIVE: $\lim_{\|x\| \to \infty} f(x) = +\infty \Rightarrow f$ IS BOUNDED FROM BELOW

$\forall k$:

$P_k(x) \geq f(x) \Rightarrow \lim_{\|x\| \to \infty} P_k(x) = +\infty$

$\Rightarrow P_k(x)$ IS COERCIVE $\Rightarrow$ HAS MINIMIZER x_k

(THESE HOLD)

CONVERGING SEQUENCE:

$f(x_k) \leq P_k(x_k) \leq f(x^0) \Rightarrow \|x_k\|$ IS BOUNDED (AS P_k IS COERCIVE)

$\Rightarrow \{x_k\}$ CONTAINS CONVERGING SUBSEQUENCE BY BOLZANO-WEIERSTRASS PROPERTY

↑

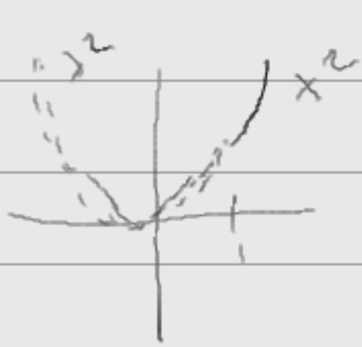
CHECK

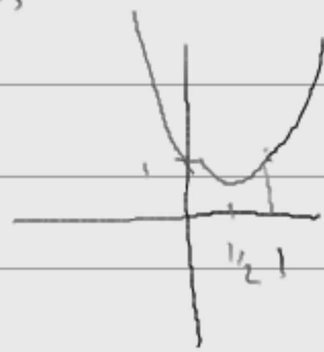
$\square$

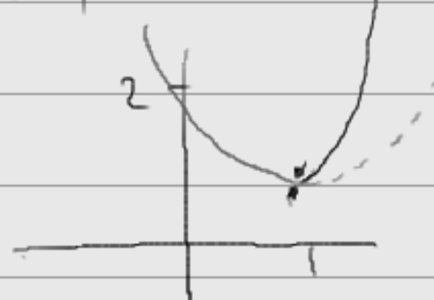
EXAMPLE (RECYCLE) - ABSOLUTE VALUE PEN. PENALTY

$$(P)\begin{cases} \text{MIN } f(x) = x^2 \\ \text{ST. } g(x) = 1 - x \le 0 \end{cases} \qquad x \in \mathbb{R}$$

$$F_K(x) = \begin{cases} x^2 + K(1-x) & x < 1 \\ x^2 & x \ge 1 \end{cases}$$







$K=0$

$K=1$ $\quad x^2 - x + 1$

$F_1'(x) = 2x - 1 = 0$

$K=2$

$x^2 - 2x + 2$

$2x - 2$

$\Rightarrow \quad X_2 = X^* = 1$

$\exists K$ ST $X_K = X^*$.

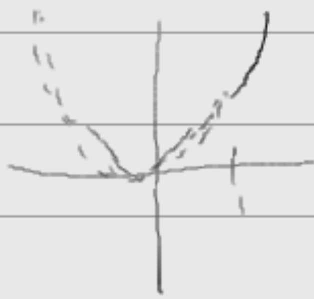
RECALL: WITH COURANT-BELTRAMI

$$X_K = \frac{K}{1+K}$$

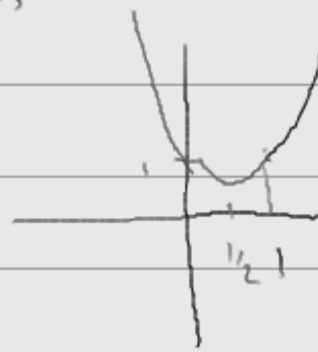
EXAMPLE (RECYCLE) - USING ABSOLUTE VALUE PENALTY

$$(P)\begin{cases} \text{MIN } f(x) = x^2 \\ \text{ST. } g(x) = 1 - x \le 0 \end{cases} \qquad x \in \mathbb{R}$$

$$F_k(x) = \begin{cases} x^2 + k(1-x) & x < 1 \\ x^2 & x \ge 1 \end{cases}$$

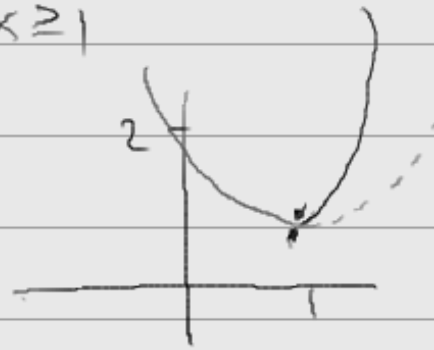


$k=0$



$k=1$ $\quad x^2 - x + 1$

$2x - 1 = 0$



$k=2$ $\quad x^2 - 2x + 2$

$2x - 2$

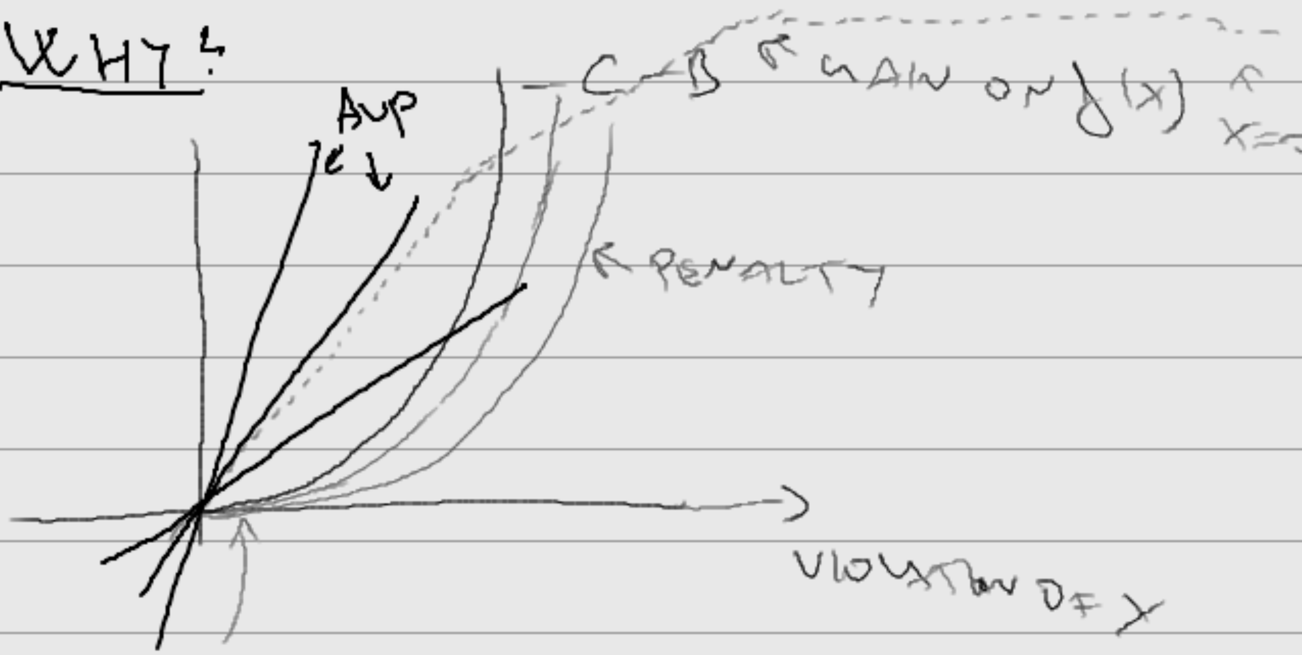
$\Rightarrow \quad X_2 = X^* = 1$

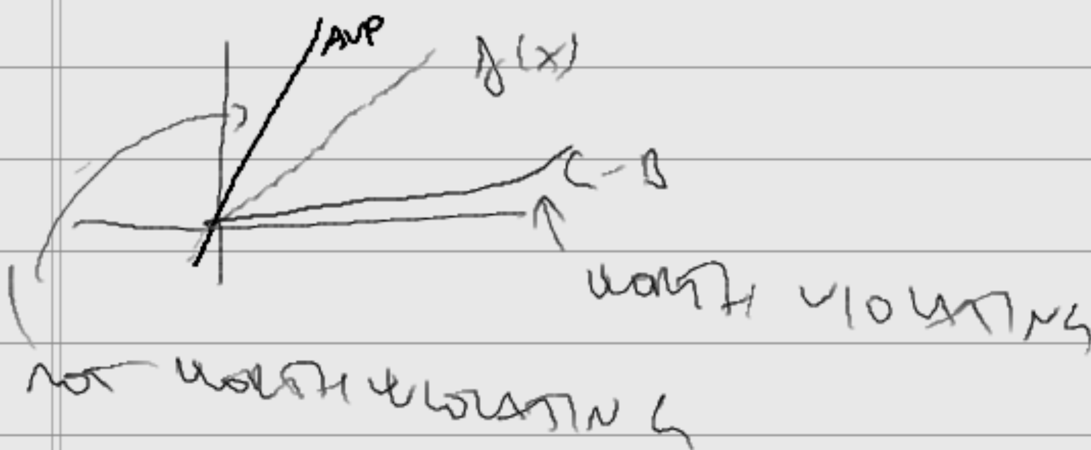
$\exists k$ ST $X_k = X^*$.

RECALL: WITH COURANT-BELTRAMI

$$X_k = \frac{k}{1+k}$$

WHY?





DEF:

PENALTY FUNCTIONS WHERE $X_K = X^0$ FOR SOME K ARE CALLED **EXACT**